Logistics

- We have assigned Grad students to papers and are now figuring out the dates.
- Include your team ID in your github readme.
- Project 1: Papers due on February 10^{th}.
- February 3rd is the drop without a 'W' grade and full refund deadline.

Logistics

- Student Presentation on Wednesday: Lorenz, E., “Computational Chaos”, *Physica D*, 1988
- I will post the presentation review forms this afternoon.

Lecture 5

Fixed Point Analysis

Basic Concepts: Higher Order Equations

Consider a system defined by a higher-order differential equation such as

$$\frac{d^3x}{dt^3} = f(x, \dot{x}, \ddot{x})$$

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We can always turn this into a system of 1st-order differential equations

Just by renaming the derivative terms:

Basic Concepts: Higher Order Equations

Consider a system defined by a higher-order differential equation such as

$$\frac{d^3x}{dt^3} = f(x, \dot{x}, \ddot{x})$$

Substitute

We can always turn this into a system of 1st-order differential equations Just by renaming the derivative terms:

$$x_1(t) = x(t)$$
$$x_2(t) = \dot{x}(t)$$
$$x_3(t) = \ddot{x}(t)$$

Basic Concepts: Higher Order Equations

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Rewrite as a system of equations

$$\frac{dx_1}{dt} = x_2$$
$$\frac{dx_2}{dt} = x_3$$
$$\frac{dx_3}{dt} = f(x_1, x_2, x_3)$$

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Basic Concepts: Higher Order Equations

This is generally true for differential equations of any order:

so let's use
a more general
and compact notation.

Basic Concepts: Higher Order Equations

$$\frac{d\vec{x}}{dt} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ f(x_1, x_2, \ldots, x_m) \end{bmatrix}$$

so let's use
a more general
and compact notation.

Basic Concepts: Higher Order Equations

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ \vdots \\ x_m(t+1) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ x_3(t) \\ \vdots \\ f(x_1(t), x_2(t), \ldots, x_m(t)) \end{bmatrix}$$

or in the case of a map.

Basic Concepts: Phase Space

Definition: The space spanned by all allowed values of $x_1 \ldots x_m$ in a system defined by:

$$\frac{d\vec{x}}{dt} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ f(x_1, x_2, \ldots, x_m) \end{bmatrix}$$

Basic Concepts: Trajectory or Orbit

Definition: A solution in the system defined by:

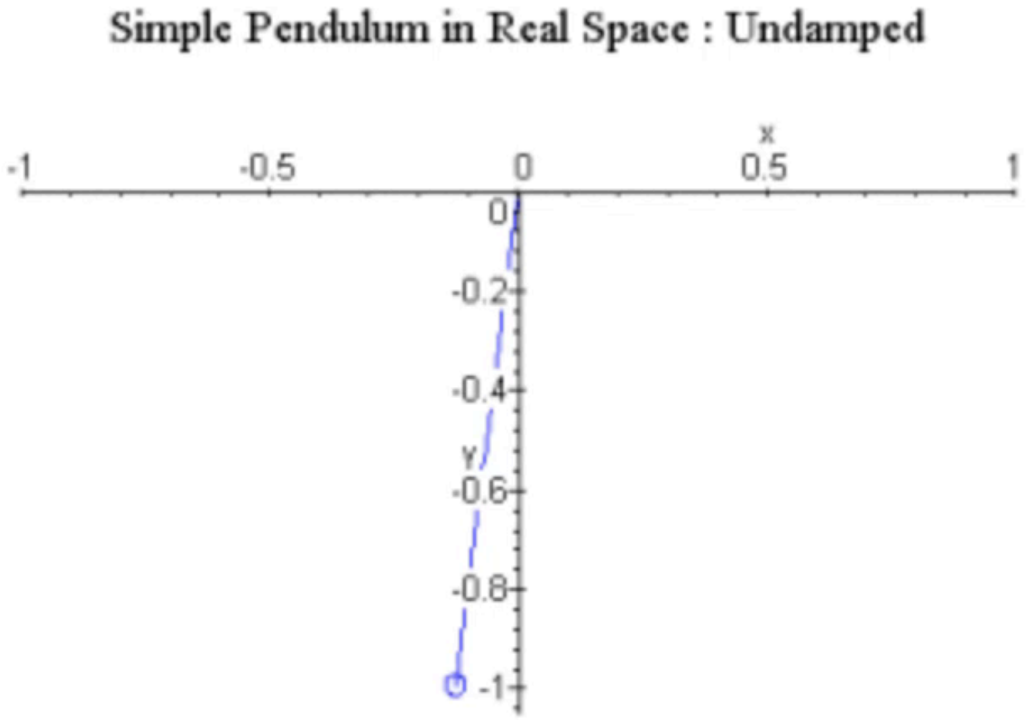
$$\frac{d\vec{x}}{dt} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ f(x_1, x_2, \ldots, x_m) \end{bmatrix}$$

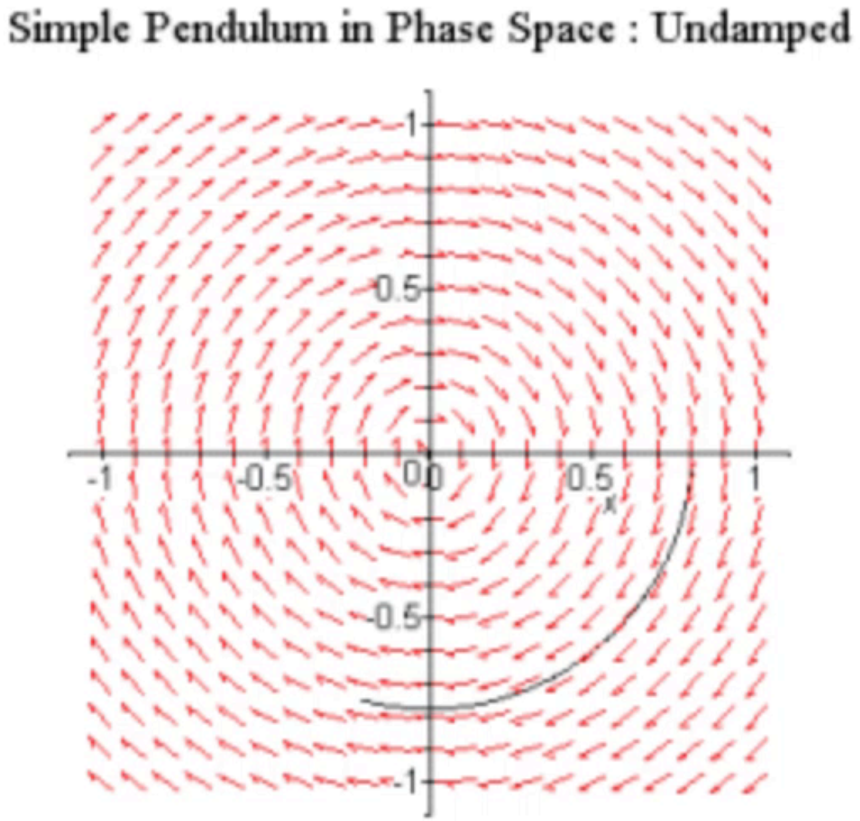
and initial conditions: $x_1 = c_1, \ldots, x_m = c_2$.

Basic Concepts: Recall from last time

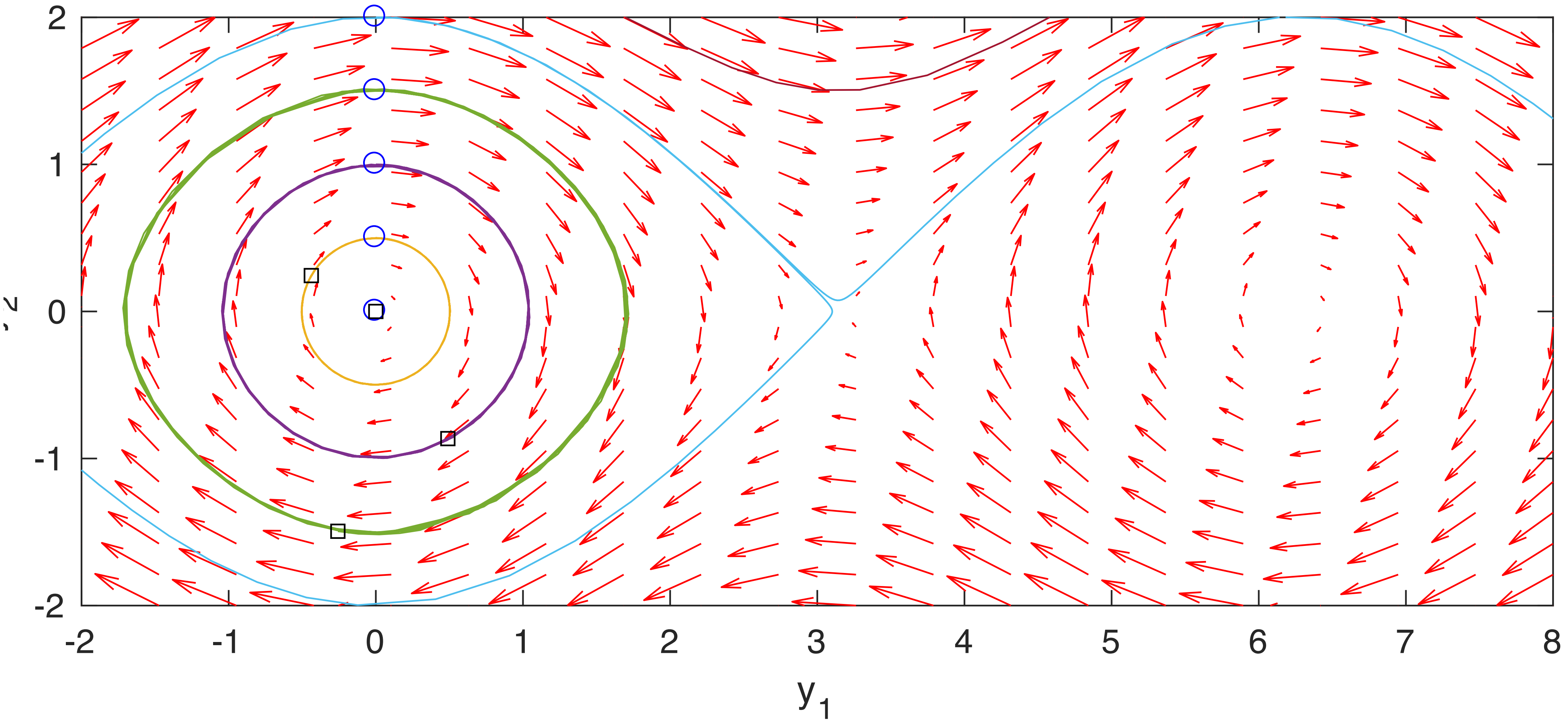
System of equations for an undamped pendulum:

$$y_1' = y_2$$
$$y_2' = -\sin(y_1)$$





Basic Concepts: What do the three fixed points mean physically?



Basic Concepts: Higher Order Equations

Since we can rewrite any higher-order
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we only have to consider 1st-order equations
from now on.

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How does this relate to part 1 of the project?

Fixed Points and Stability

Consider a fixed point x^* for a 1D system:

$$\dot{x} = f(x)$$

$$f(x^*) = 0$$

Fixed Points and Stability

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The fixed point is the value that keeps the function at 0.

The stability of x^* depends on the direction of the flow nearby.

Fixed Points and Stability

Why?

Consider a fixed point x^* for a 1D system:

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Fixed Points and Stability

The stability of x^* depends on the direction of the flow nearby.

To find the direction near x^* we linearise $\dot{x}$ near x^*

Fixed Points and Stability

To find the direction near x^*

we linearise $\dot{x}$ near x^*

Recall from Calculus I the Taylor expansion:

$$\frac{d}{dt}(x - x^*) = \dot{x} \approx f(x^*) + f'(x^*)(x - x^*) + \ldots$$

Fixed Points: Classification

Our system: $\dot{x} = f(x), f(x^*) = 0$

Plugging in and neglecting higher order terms:

$$\frac{d}{dt}(x - x^*) = \dot{x} \approx f(x^*) + f'(x^*)(x - x^*) + \dots$$

$$\frac{d}{dt}(x - x^*) \approx 0 + f(x^*)(x - x^*)$$

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$$\frac{d}{dt}(x - x^*) \approx 0 + f(x^*)(x - x^*)$$

Fixed Points: Classification

Solving: $\frac{d}{dt}(x - x^*) \approx f'(x^*)(x - x^*)$

Relabel: $\Delta x := (x - x^*)$

Solve: $\frac{d\Delta x}{dt} \approx f'(x^*)\Delta x$

Fixed Points: Classification

$$\text{Solve: } \frac{d\Delta x}{dt} \approx f'(x^*)\Delta x$$

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Fixed Points: Classification

$$\text{Solve:}\quad \frac{d\Delta x}{dt} \approx f'(x^*)\Delta x$$

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$$\frac{d\Delta x}{\Delta x} \approx f'(x^*)dt$$

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$$\int \frac{d\Delta x}{\Delta x} \approx \int f'(x^*)dt$$

Fixed Points: Classification

Solve: $$\frac{d\Delta x}{dt} \approx f'(x^*)\Delta x$$

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$$\frac{d\Delta x}{\Delta x} \approx f'(x^*)dt$$

$$\int \frac{d\Delta x}{\Delta x} \approx \int f'(x^*)dt$$

$$\int \frac{1}{\Delta x} d\Delta x \approx \int f'(x^*)dt$$

Fixed Points: Classification

Solve: $$\frac{d\Delta x}{dt} \approx f'(x^*)\Delta x$$

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$$\ln \Delta x \approx f'(x^*)t$$

$$\Delta x \approx e^{f'(x^*)t}$$

Fixed Points: Classification

Solve: $$\frac{d\Delta x}{dt} \approx f'(x^*)\Delta x$$

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$$\boxed{\Delta x \approx e^{f'(x^*)t}}$$

Fixed Points: Classification

The flow near fixed point, x^*, for $\dot{x} = f(x)$ with $f(x^*) = 0$ is

$$\Delta x \approx e^{f'(x^*)t}$$

Fixed Points: Classification

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What do points Δx distance from x^*

do as time gets large?

Fixed Points: Classification

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What do points Δx distance from x^* do as time gets large?

$$\lim_{t\to\infty} e^{f'(x^*)t} = \begin{cases} 0 & \text{if } f'(x^*) < 0 \\ \infty & \text{if } f'(x^*) > 0 \end{cases}$$

Fixed Points: Classification

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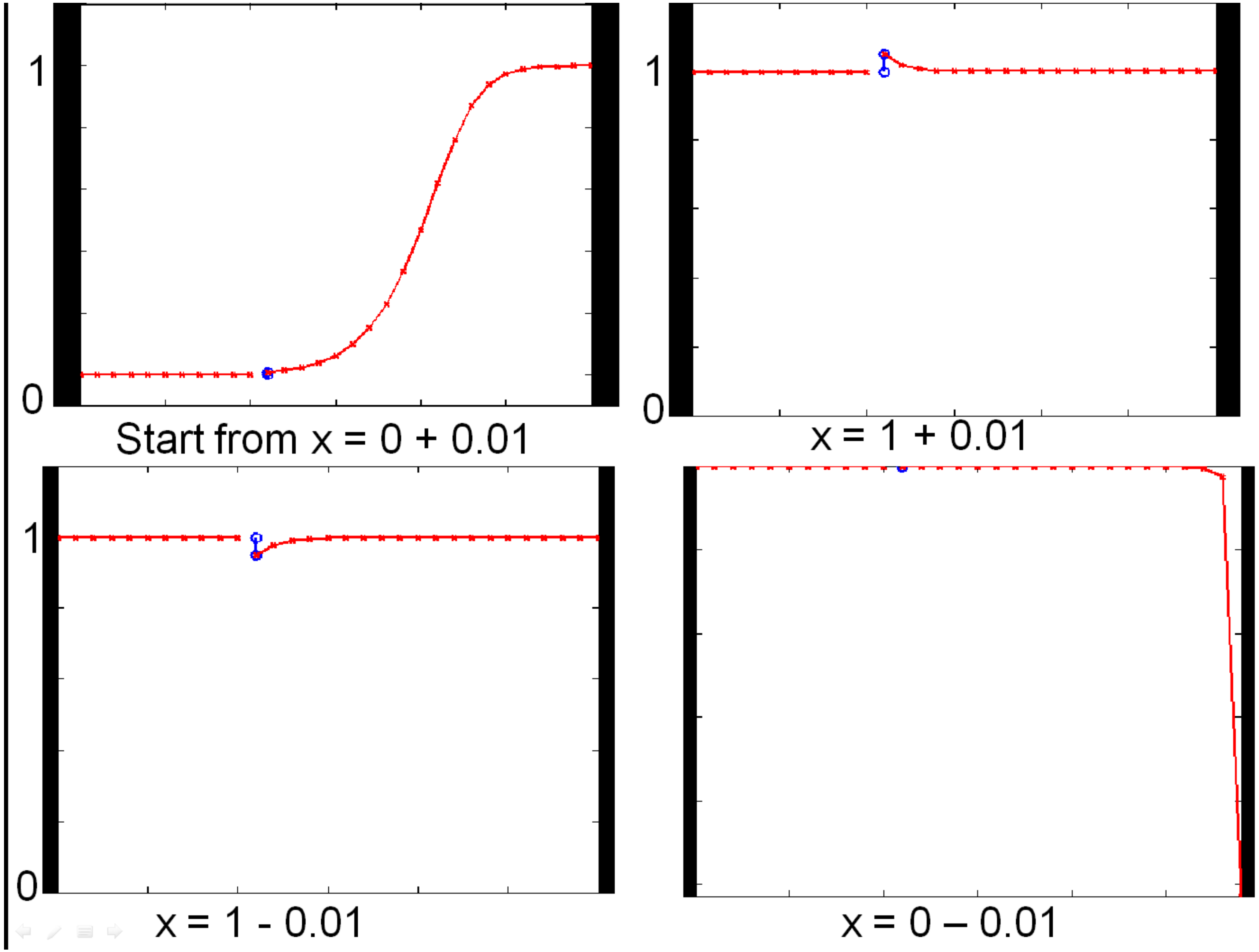
So for a 1D system the flow diverges
away from the fixed point when the derivative
of points near the fixed point are negative and converge
on the fixed point if positive.

Fixed point at 0 is unstable.



Fixed point at 1 is stable.

Lyapunov Exponent

Aleksandr Lyapunov, Russia, turn of the 20th Century.

Fixed Points: Lyapunov Exponent

$$\lim_{t\to\infty} e^{f'(x^*)t} = \begin{cases} 0 & \text{if } f'(x^*) < 0 \\ \infty & \text{if } f'(x^*) > 0 \end{cases}$$

As we just showed the time evolution close to a fixed point x^* is generally exponential: $\Delta x = e^{\lambda t}$, where $\boxed{\lambda} = f'(x^*)$.

Lyapunov Exponent

Fixed Points: Lyapunov Exponent

A negative Lyapunov Exponent $\implies$ the flow moves exponentially towards the fixed point.

A positive Lyapunov Exponent $\implies$ the flow moves exponentially away from the fixed point.

Fixed Points: Discrete Maps

$$x(t+1) = f(x(t))$$

$$x(t+1) = f(x(t)) \approx f(x^*) + g'(x^*)(x(t) - x^*)$$

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For discrete maps the fixed point is at:

$$f(x^*) = x^*$$

Fixed Points: Discrete Maps

Why?

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Fixed Points: Discrete Maps

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Fixed Points: Discrete Maps

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$$\Delta x(t) = e^{\lambda t}$$

Fixed Points: Discrete Maps

Solve for λ

$$\Delta x(t+1) = x(t+1) - x^* = f'(x^*)\Delta x(t)$$

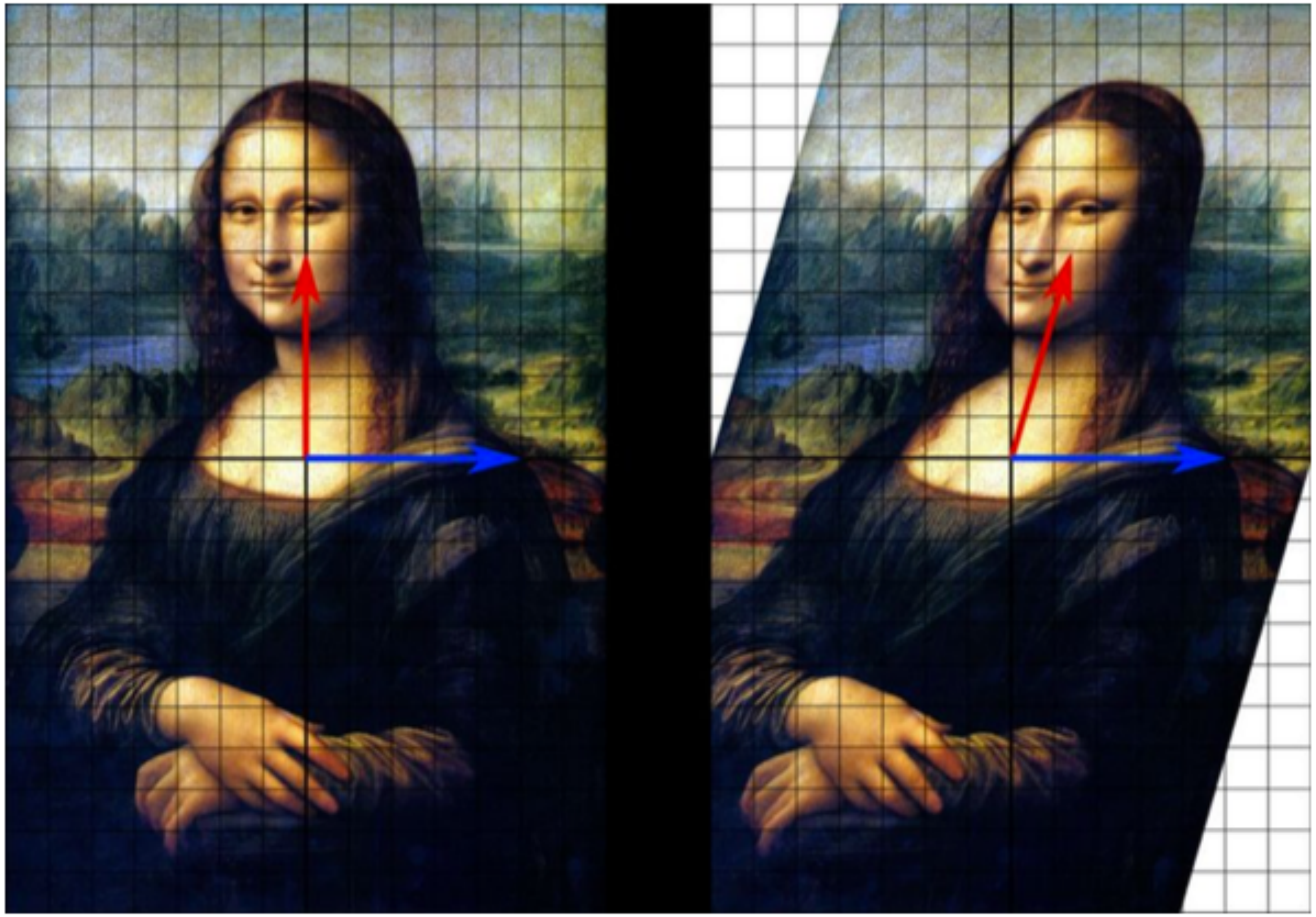
$$\Delta x(t+1) = x(t+1) - x^* = f'(x^*)\Delta x(t)$$

$$\Delta x(t) = e^{\lambda t}$$

The Lyapunov exponent for maps is:

$$\lambda = \ln|f'(x^*)| = \begin{cases} < 0 \text{ if } |f'(x^*)| < 1 \\ > 0 \text{ if } |f'(x^*)| > 1 \end{cases}$$

Fixed Points: Multidimensional Systems



Wikipedia, 2017

The blue line is an eigenvector.

It doesn't change as we transform the image.

Fixed Point Analysis: Multidimensional Systems

Consider the 2D system:

$$\dot{x} = y$$

$$\dot{y} = 2x + y$$

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Fixed Point Analysis: Multidimensional Systems

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In matrix form:

$$\begin{bmatrix} 0x + 1y \\ 2x + 1y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Fixed Point Analysis: Multidimensional Systems

Fixed points are where:

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In matrix form:

$$\begin{bmatrix} 0x + 1y \\ 2x + 1y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Fixed Point Analysis: Multidimensional Systems

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The eigenvalues describe the flow nearby

Fixed Point Analysis: Multidimensional Systems

An enormous shortcut . . .

Fixed points are where:

$$y = 0$$

$$2x + y = 0$$

In matrix form:

$$\begin{bmatrix} 0x + 1y \\ 2x + 1y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The eigenvalues describe the flow nearby

```
>> M = [0,1;2,1]

M =

     0     1
     2     1

>> eig(M)

ans =

    -1
     2
```

Fixed Point Analysis: Multidimensional Systems

The eigenvalues describe
the flow nearby...

...if the system is LINEAR (or nearly so).
If non-linear we need to take the eigenvalues
of the **Jacobian** matrix (see Lecture 7).

Jacobian

Given a set of equations:

$$y_1 = f_1(x_1, x_2, \ldots, x_n)$$

$$y_2 = f_2(x_1, x_2, \ldots, x_n)$$

$$\vdots$$

$$y_m = f_m(x_1, x_2, \ldots, x_n)$$

Jacobian

The Jacobian is:

The partial derivatives for each equation and in each direction.

(Calc III)

$$\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ & \vdots & & \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$

Example: Holmes Map

$$x_{t+1} = y_t$$

$$y_{t+1} = -bx_t + ay_t - y_t^3$$

Example: Holmes Map

$$x_{t+1} = y_t$$

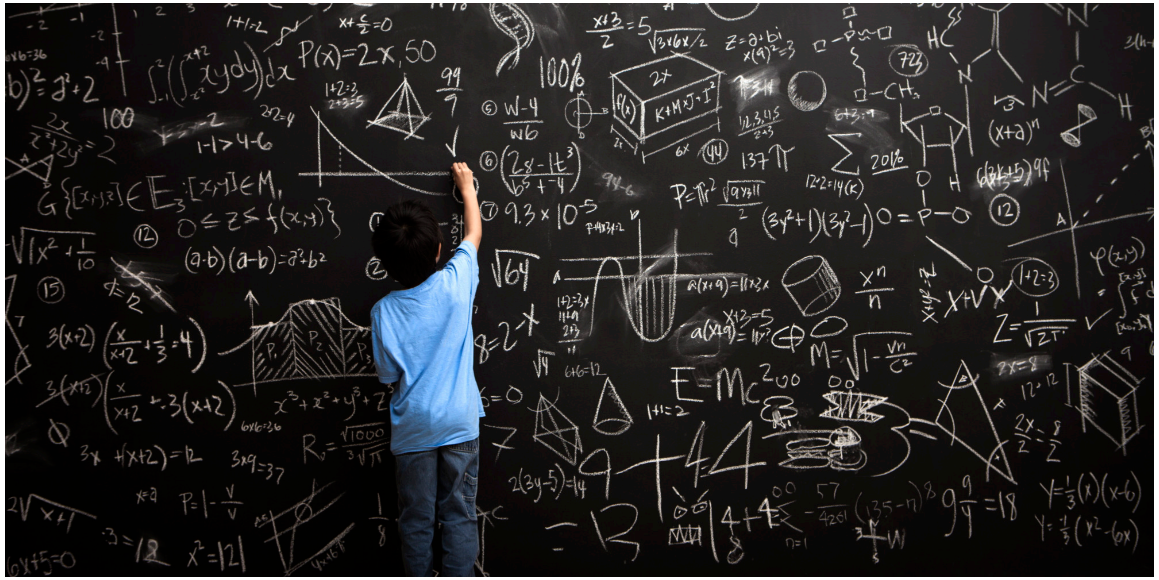
$$y_{t+1} = -bx_t + ay_t - y_t^3$$

$$J = \begin{bmatrix} \dfrac{\partial \overset{x_{t+1}}{\boxed{y_t}}}{\partial x_t} & \dfrac{\partial \overset{x_{t+1}}{\boxed{y_t}}}{\partial y_t} \\ \dfrac{\partial(-bx_t + ay_t - y_t^3)}{\partial x_t} & \dfrac{\partial(-bx_t + ay_t - y_t^3)}{\partial y_t} \end{bmatrix}$$

Example: Holmes Map

$$J = \begin{bmatrix} \dfrac{\partial y_t}{\partial x_t} & \dfrac{\partial y_t}{\partial y_t} \\ \dfrac{\partial(-bx_t + ay_t - y_t^3)}{\partial x_t} & \dfrac{\partial(-bx_t + ay_t - y_t^3)}{\partial y_t} \end{bmatrix}$$

Let’s Calculate it…



Example: Holmes Map

$$J = \begin{bmatrix} \frac{\partial y_t}{\partial x_t} & \frac{\partial y_t}{\partial y_t} \\ \frac{\partial(-bx_t + ay_t - y_t^3)}{\partial x_t} & \frac{\partial(-bx_t + ay_t - y_t^3)}{\partial y_t} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -b & a - 3y^2 \end{bmatrix}$$

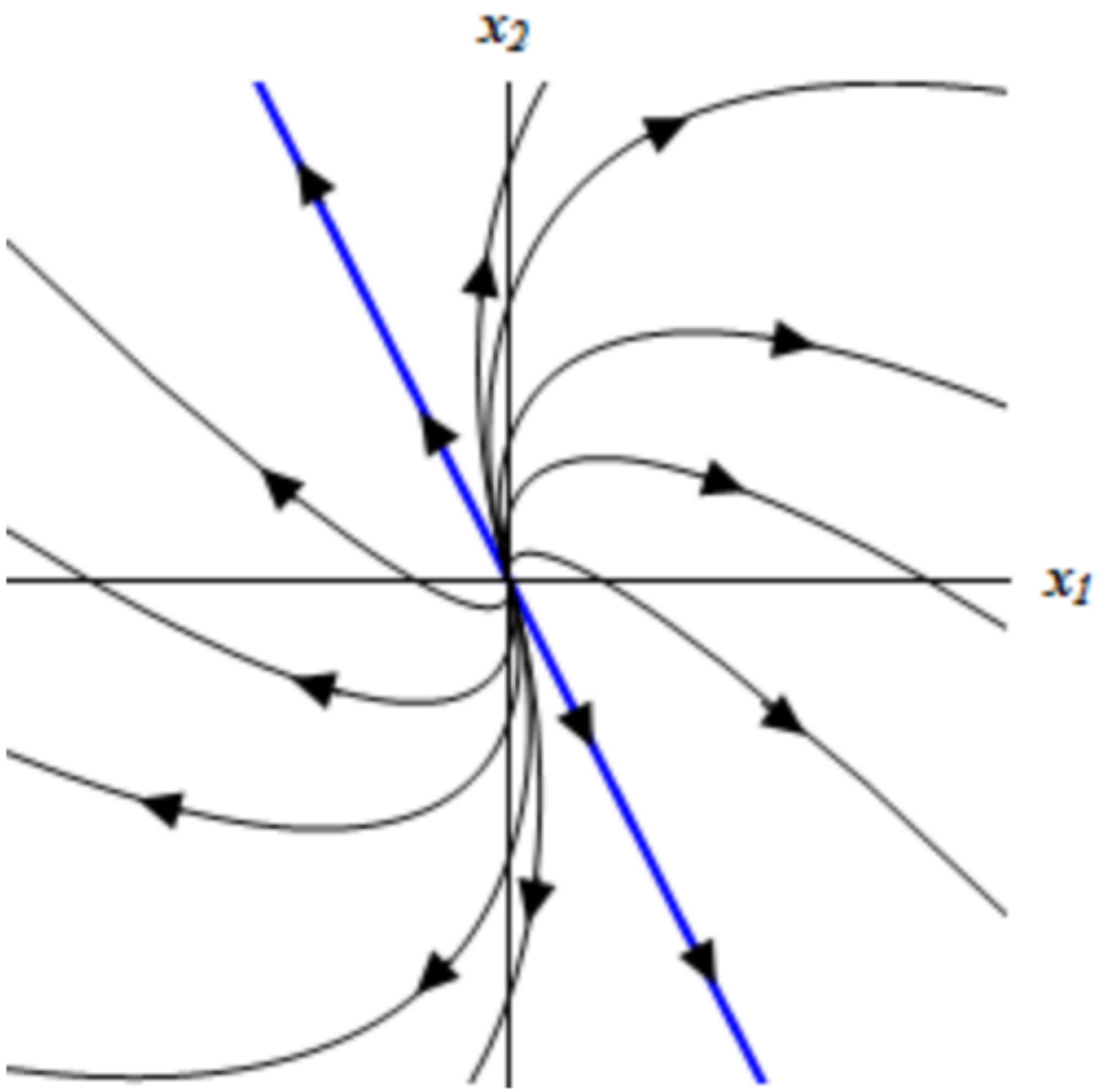
In Matlab:

```
>> syms x y a b
>> HolmesMap = [y; -b*x+a*y-y^3]
HolmesMap = y - y^3 + a*y - b*x

>> HolmesMapJ = jacobian(HolmesMap, [x,y])
HolmesMapJ = [0,              1]
             [-b, - 3*y^2 + a]

>> eig(HolmesMapJ)
ans =  a/2 - (a^2 - 6*a*y^2 + 9*y^4 - 4*b)^(1/2)/2 - (3*y^2)/2
       a/2 + (a^2 - 6*a*y^2 + 9*y^4 - 4*b)^(1/2)/2 - (3*y^2)/2
```

Classifying Fixed Points (2D Systems)

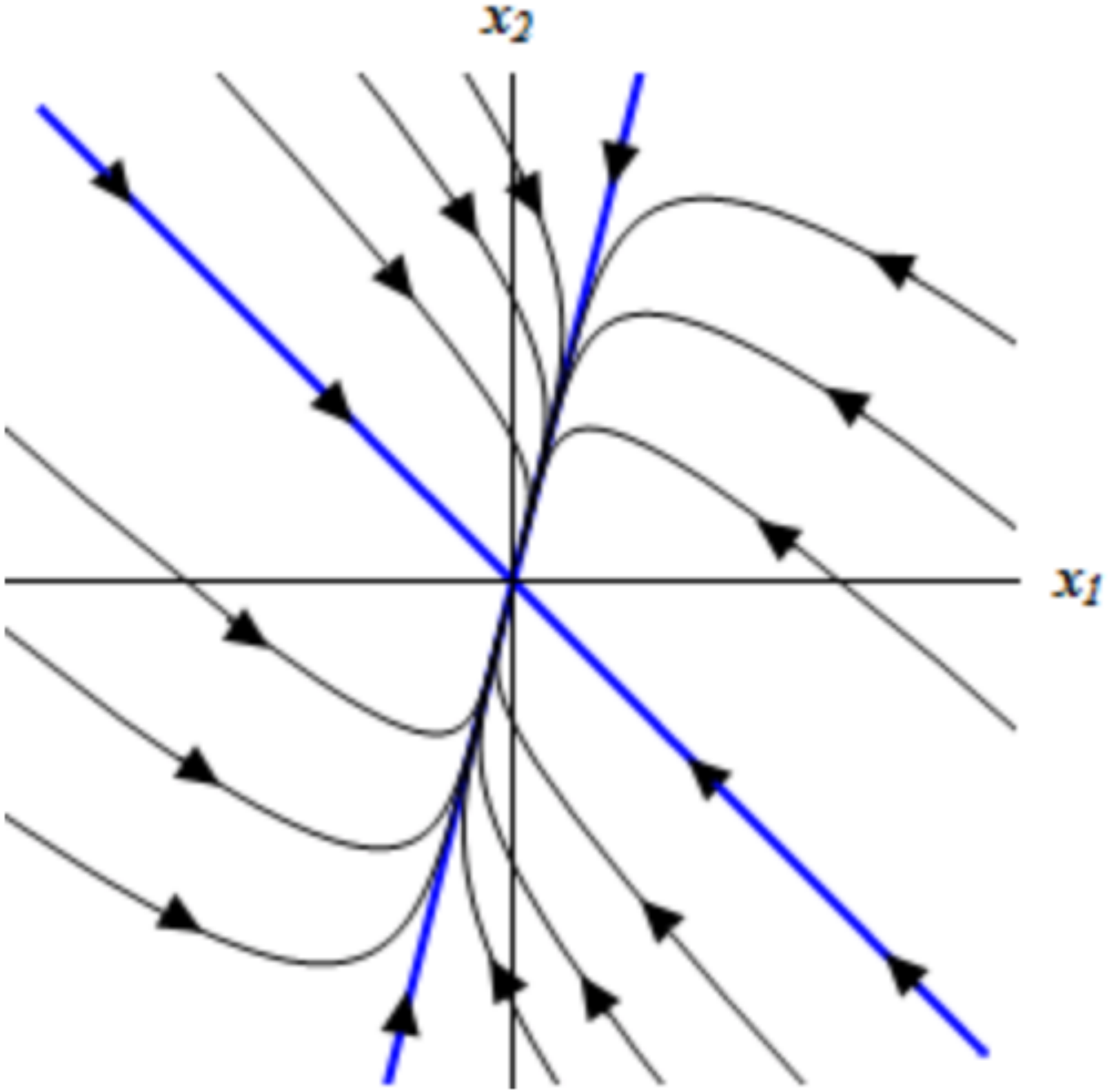
Eigenvalues	Stability	Name
Real and positive	Unstable	Source
Real and negative	Stable	Sink
Real mixed signs	Unstable	Saddle point
Complex with positive real part	Unstable	Spiral Source
Complex with negative real part	Stable	Spiral Sink
Imaginary	Unstable	Center

Classifying Fixed-Points (2D)



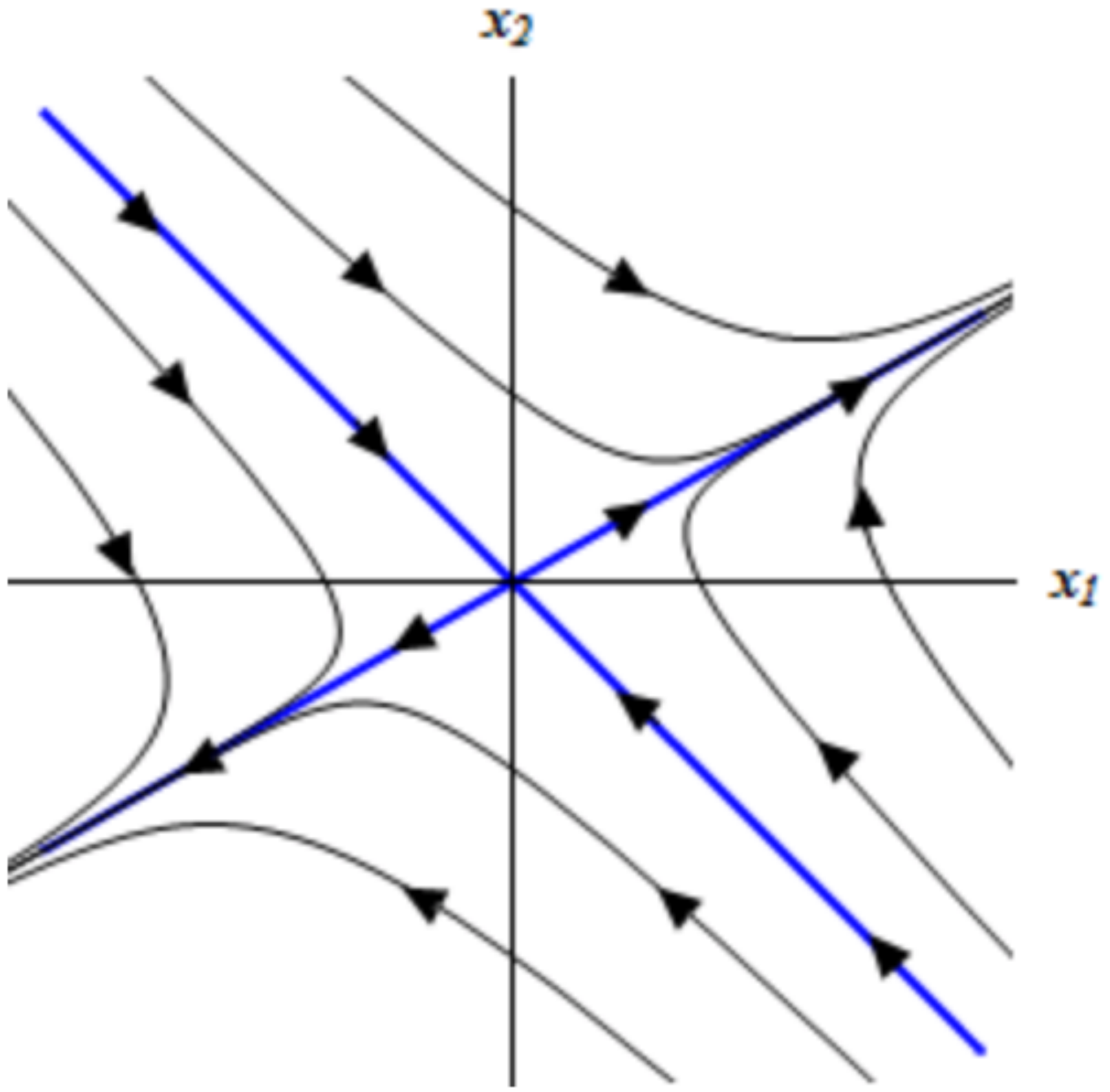
Unstable fixed-point (source)

Classifying Fixed-Points (2D)



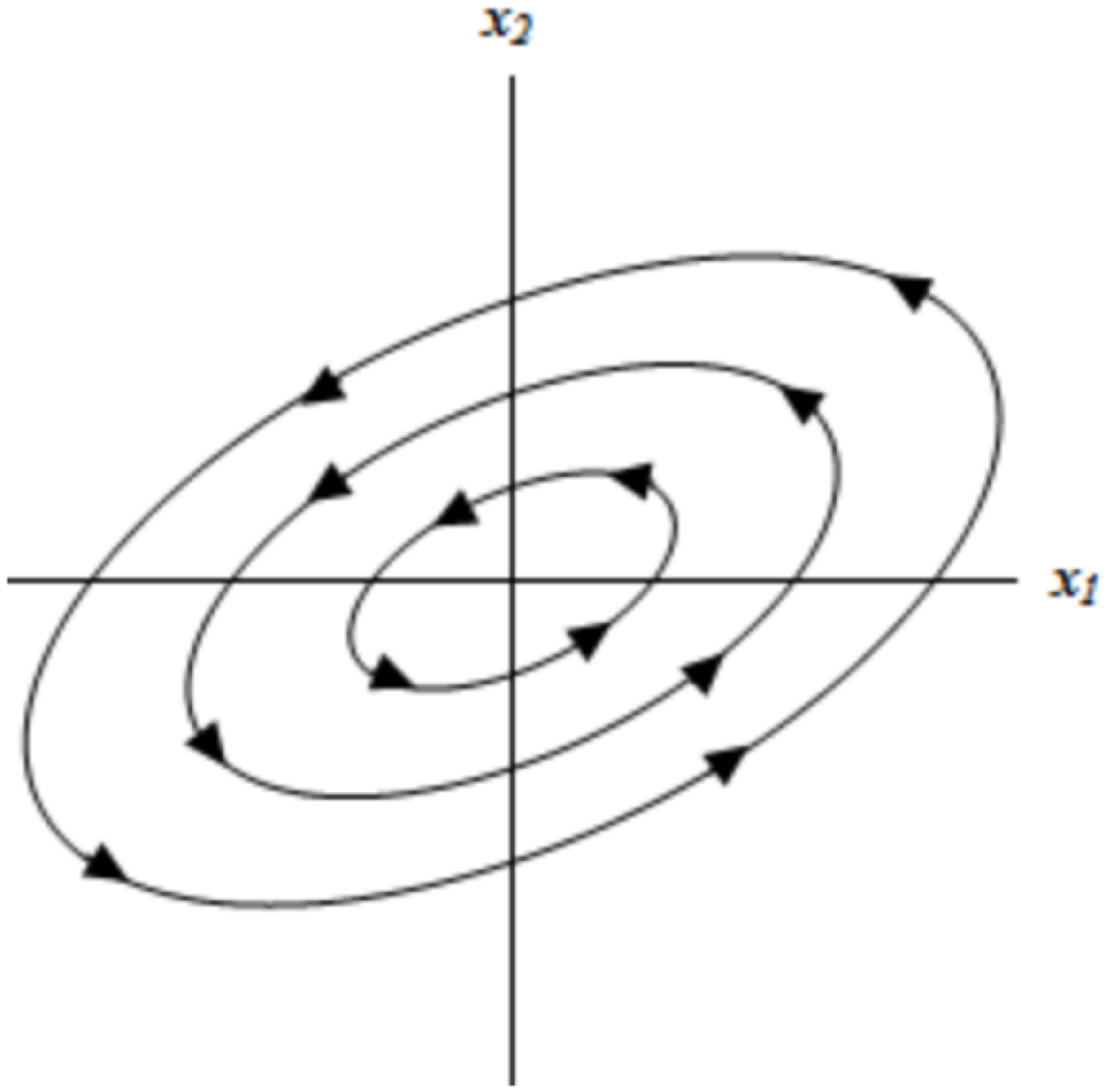
Stable fixed-point (sink)

Classifying Fixed-Points (2D)



Saddle point

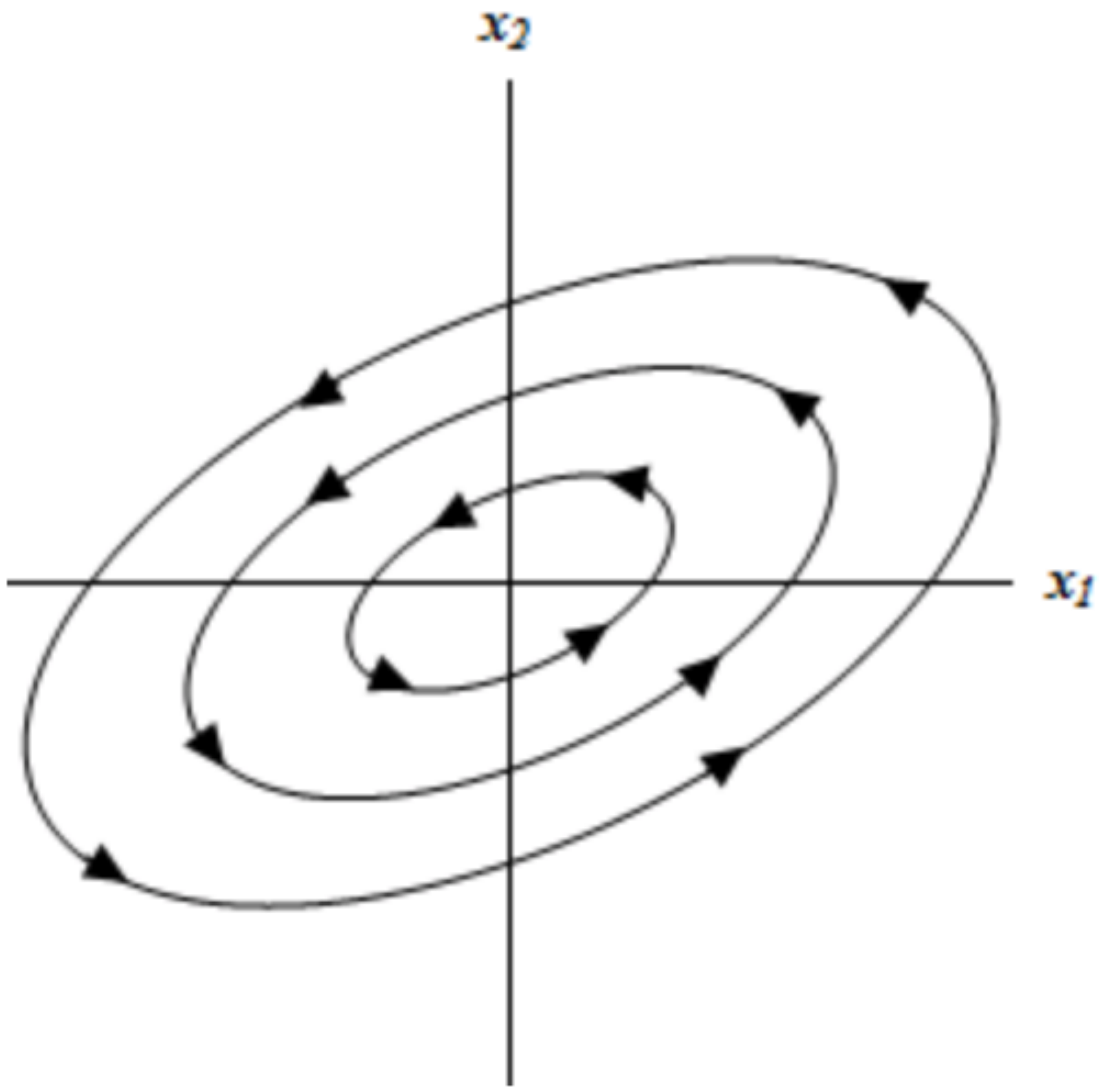
Classifying Fixed-Points (2D)



Center

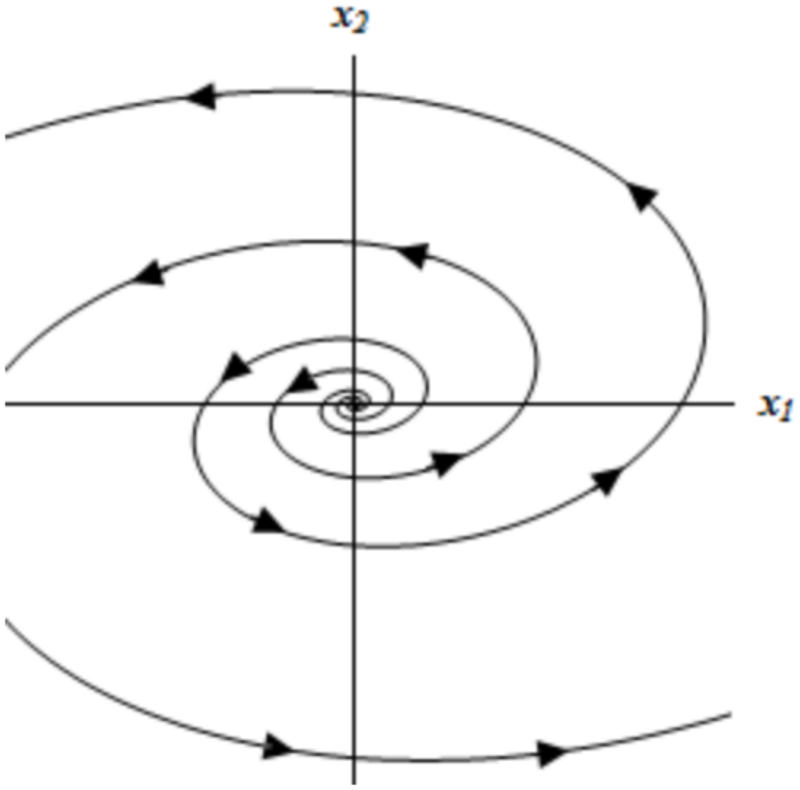
Classifying Fixed-Points (2D)

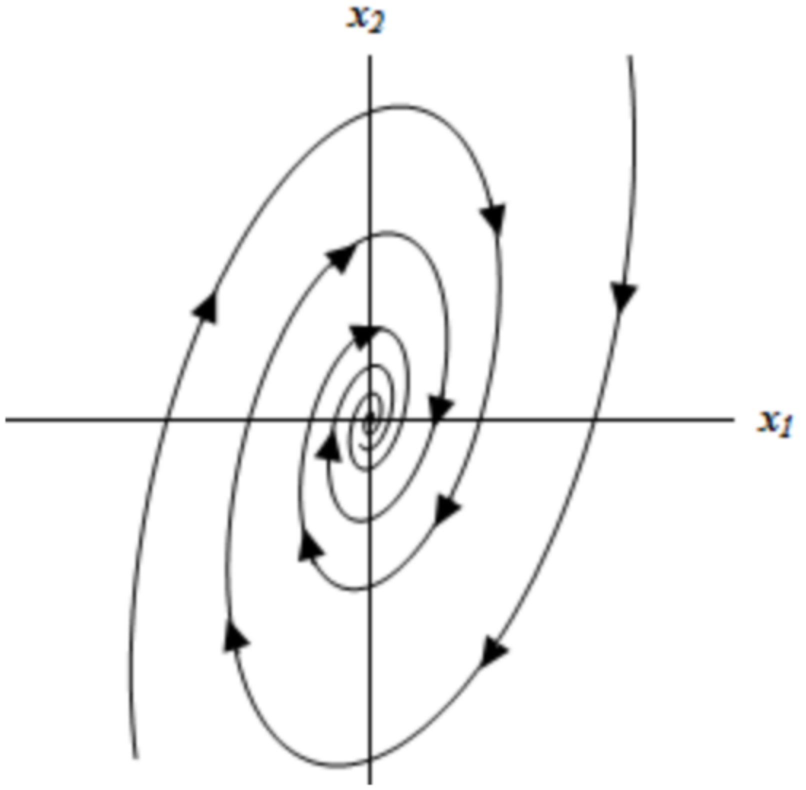
Center



Recall the Lotke-Volterra Model

Classifying Fixed-Points (2D)





Spirals (stable and unstable)

Lecture 5

Ergodicity

Ergodicity

A dynamical system in which trajectories come arbitrarily close to any point in the phase space no matter the initial conditions.

This implies that the time average is equal to the spacial average.

Basic Concepts: Ergodicity

This implies that the time average is equal to the spacial average.

So what?

Basic Concepts: Ergodicity

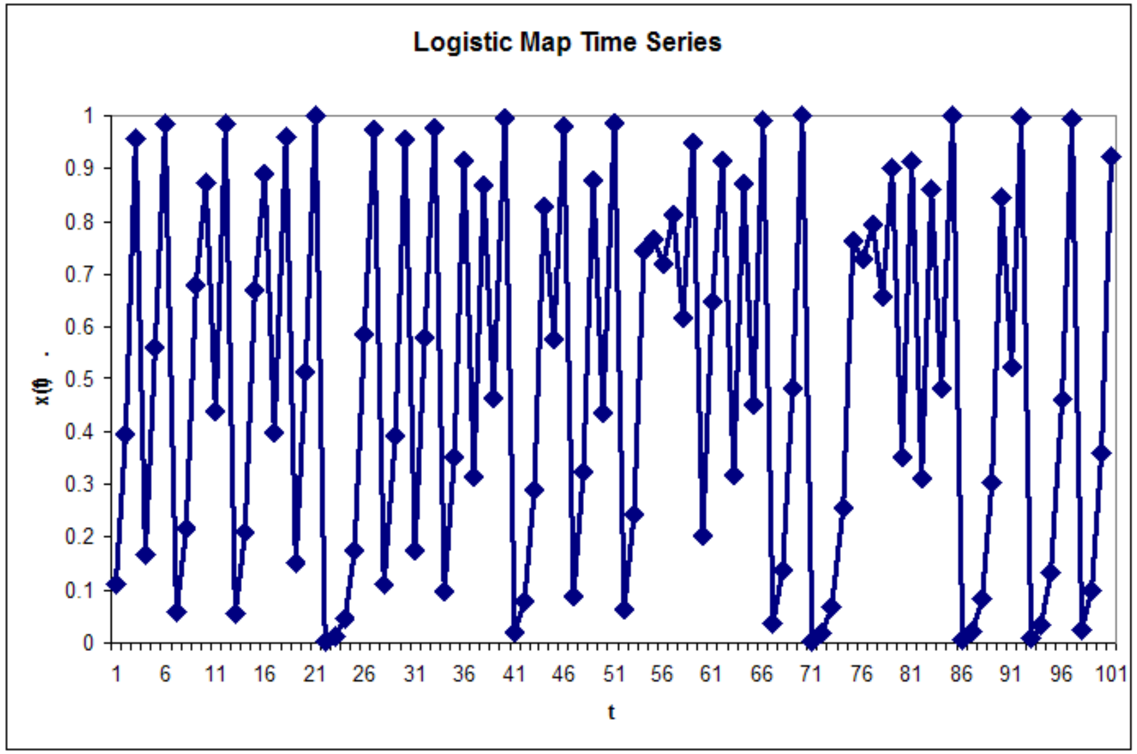
This implies that the time average is equal to the spacial average.

So what?

If a system is ergodic we can make good long-term average predictions even when the system is chaotic.

Recall the chaos of the logistic map

$$x_{t+1} = rx_t(1 - x_t)$$



The logistic map turns out to be ergodic

Ergodicity: Logistic Map

Histogram of 4000 randomly chosen initial conditions.

Histogram of time steps > 2000 for a single initial condition.

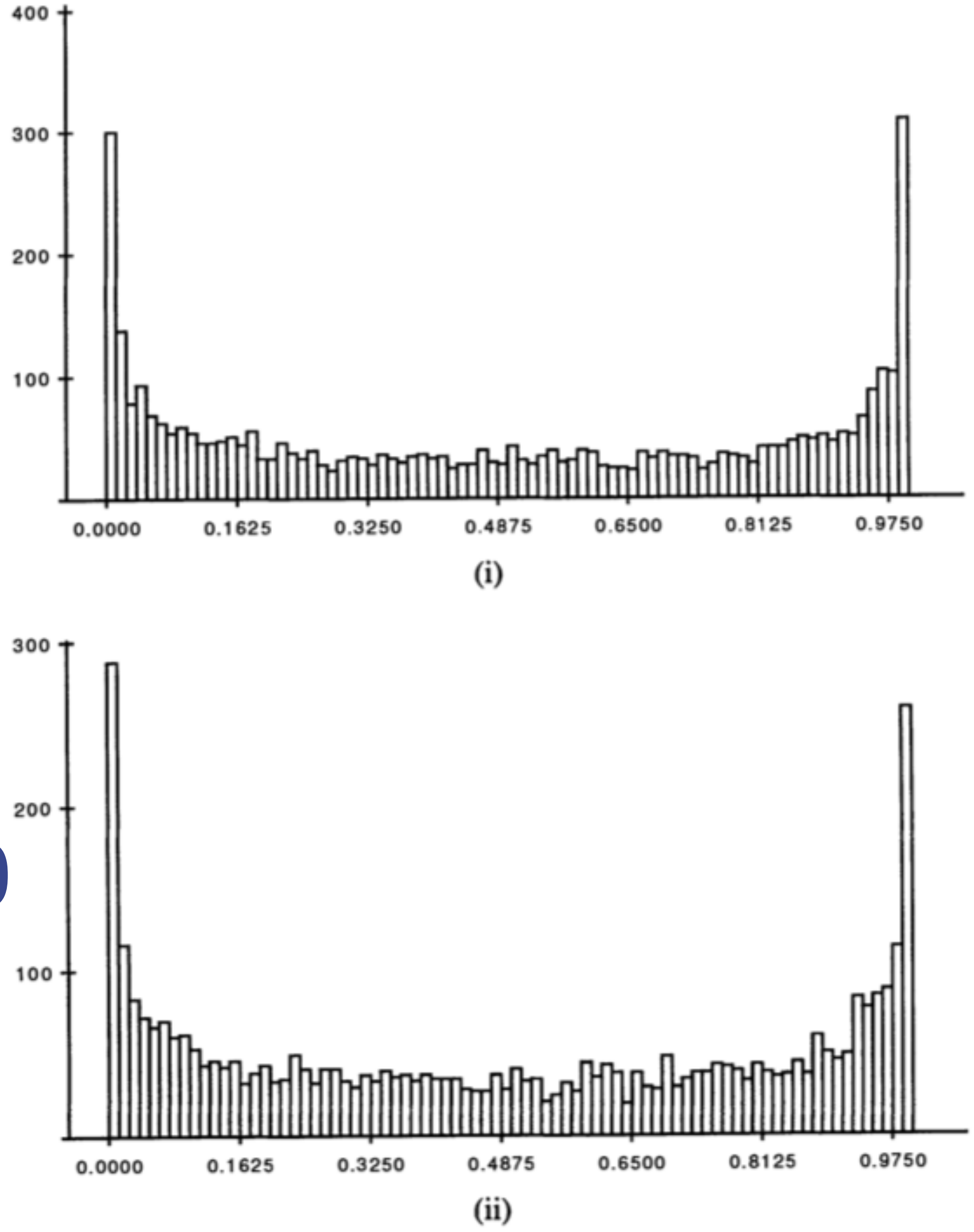
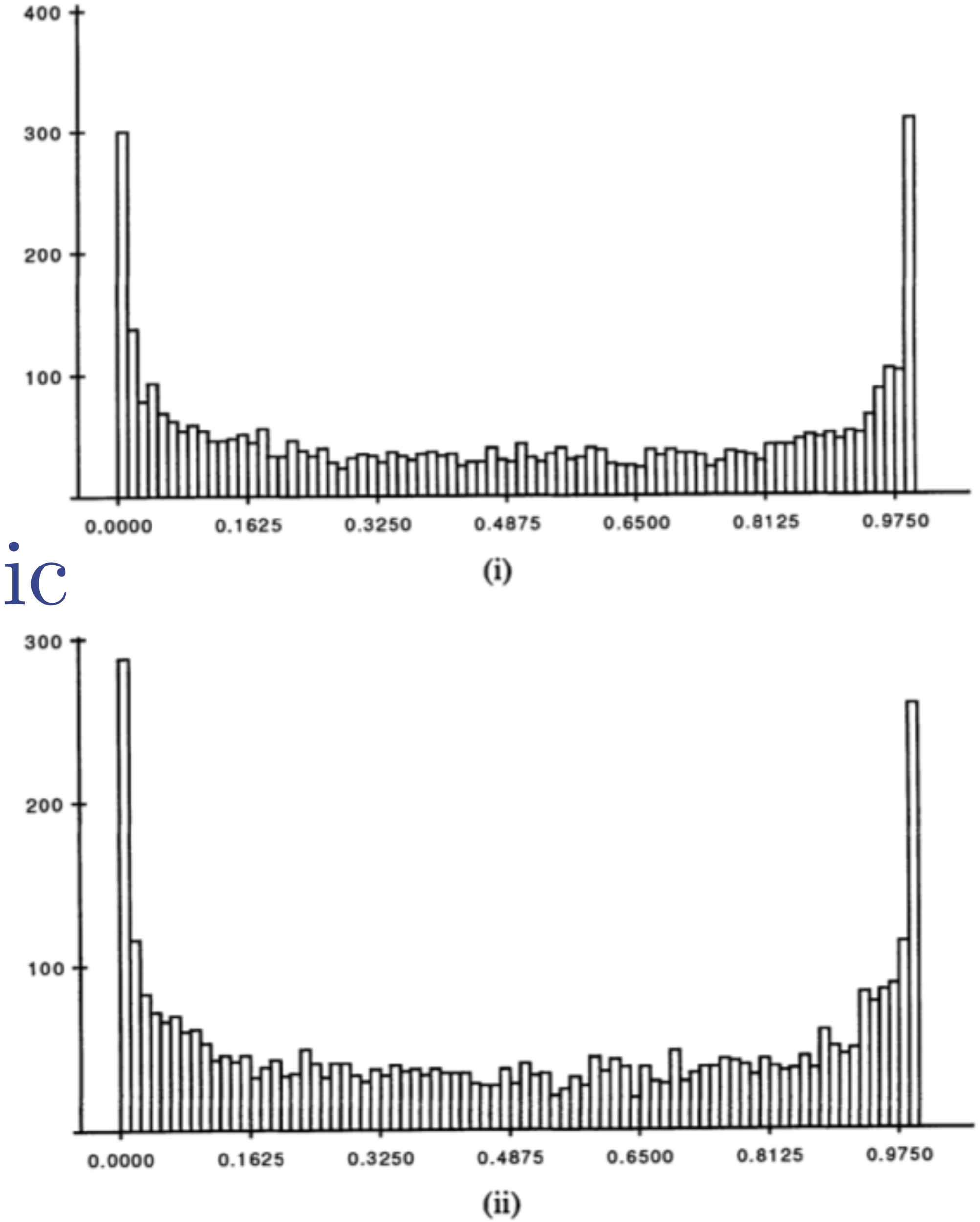


FIG. 8. *Example of ergodic behavior: Logistic map, $a = 4.0$. (i) Histogram of 4000 iterates of $x_0 = .20005$. (ii) Histogram of the logistic map at time 2000 for 4000 x_0's in* [.10005, .30005].

L Berliner, Statistics, Probability and Chaos, *Statistical Science*, 1992, Vol. 7, No. 1, 69-122

Ergodicity: Logistic Map

Because the Logistic map is ergodic we can predict its average behaviour.



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Ergodicity: Logistic Map

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So can an ergodic system have attractors?

Is the logistic map for r=1 ergodic?