

Quotations of the day:

"A mathematician is a device for turning coffee into theorems."

- Paul Erdős 1913-1996

"I have not failed. I've just found 10,000 ways that won't work."

- Thomas Edison 1847-1931

"A little inaccuracy sometimes saves a ton of explanation. "

- H. H. Munro (Saki) 1870-1916

Review:

Contrapositive

Vacuous Truth

Predicate ($P(x)$, $Q(x)$, etc)

Premise

Hypothesis

Inference Rules

- Definition:
 - A sequence of statements connected by $\wedge$
 - The last statement is the **conclusion**
 - The other statements are the **premises**
- Valid argument:
 - if the premises are true, the conclusion is also true
 - this must be the case for any particular set of statements substituted for the variables in the premises
 - this substitution process is call **instantiation**

Anatomy of an Inference Rule

Same as $(p \vee q \Rightarrow p) \wedge q \Rightarrow p$

Major Premise: **$p \vee q \Rightarrow p$**

Minor Premise: **q**

Conclusion: **$\therefore p$**

Variables: **p** and **q** , note that q
is a premise *and* a variable.

Definition: An inference rule of this form with two premises followed by a conclusion is called a ***syllogism***.

Some possible inference rules:

$p \Rightarrow q$	If stocks go up I make money.
q	I have made money.
$\therefore p$	$\therefore$ Stocks have gone up.

Converse Error!

$p \Rightarrow q$	If I study I will pass the class.
$\sim p$	I have not studied.
$\therefore \sim q$	$\therefore$ I will not pass the class.

Inverse Error!

Testing An Argument

- Is an argument valid or invalid? One test is:
 - construct a truth table for the premises and the conclusion
 - find the **critical rows** -- those in which all of the premises are true
 - check the value of the conclusion in these rows
 - if true for all critical rows, the argument is **VALID**
 - otherwise the argument is **INVALID**

Invalid Inference Rules: Converse Error.

$p \Rightarrow q$

If stocks go down I lose money.

q

I have lost money.

$\therefore p$

$\therefore$ Stocks have gone down.

Variables		/----- Premises -----\ -----\		Conclusion
p	q	q	$p \Rightarrow q$	p
T	T	T	T	T
T	F	F	F	T
F	T	T	T (vacuous)	F
F	F	F	T (vacuous)	F

Invalid Inference Rules: Inverse Error.

$p \Rightarrow q$ If I study I will pass the class.
 $\sim p$ I have not studied.
 $\therefore \sim q$ $\therefore$ I will not pass the class.

Variables /----- Premises -----\ Conclusion

p	q	$\sim p$	$p \Rightarrow q$	$\sim q$
T	T	F	T	F
T	F	F	F	T
F	T	T	T (vacuous)	F
F	F	T	T (vacuous)	T

Valid Inference Rules

- Modus Ponens
- Modus Tollens
- Disjunctive Addition
- Conjunctive Addition
- Conjunctive Simplification
- Disjunctive Syllogism
- Hypothetical Syllogism
- Contradiction Rule
- Dilemma

Modus Ponens

(the method of affirming)

Arbitrary Form

Instantiated Form

$p \Rightarrow q$

If you tickle him, he will laugh.

p

He is being tickled.

$\therefore q$

$\therefore$ He is laughing.

Valid Inference Rules: Modus Ponens

$p \Rightarrow q$ If you tickle him, he will laugh.

p He is being tickled.

$\therefore q$ $\therefore$ He is laughing.

p	q	$p \rightarrow q$	p	q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

Modus Tollens

(the method of denying)

Arbitrary Form

Instantiated Form

$p \Rightarrow q$

If you build it, they will come.

$\sim q$

They did not come.

$\therefore \sim p$

$\therefore$ You did not build it.

Valid Inference Rules: Modus Tollens

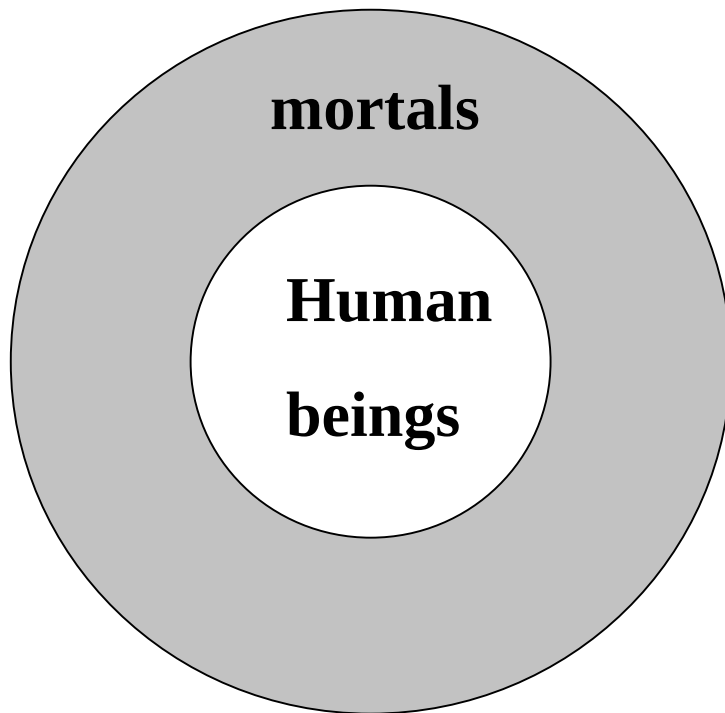
(Verifying the rule with Venn Diagrams)

Arbitrary Form	Instantiated Form	Structure
$p \Rightarrow q$	Humans are mortal.	Major Premise
$\sim q$	Zeus is not mortal.	Minor Premise
$\therefore \sim p$	$\therefore$ Zeus is not human.	Conclusion

Valid Inference Rules: Modus Tollens

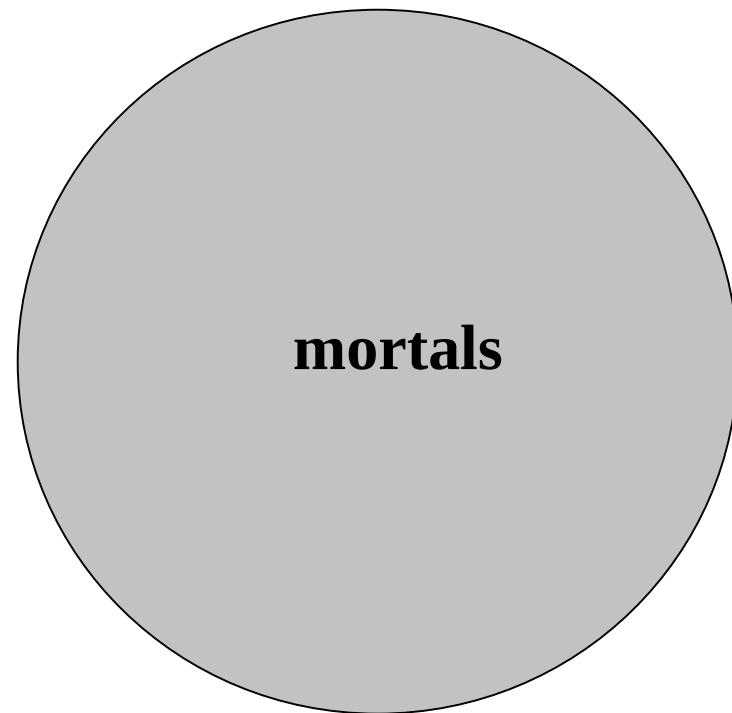
(Verifying the rule with Venn Diagrams)

Major Premise



Minor Premise

• **Zeus**



Valid Inference Rules: Modus Tollens

(Verifying the rule with Venn Diagrams)

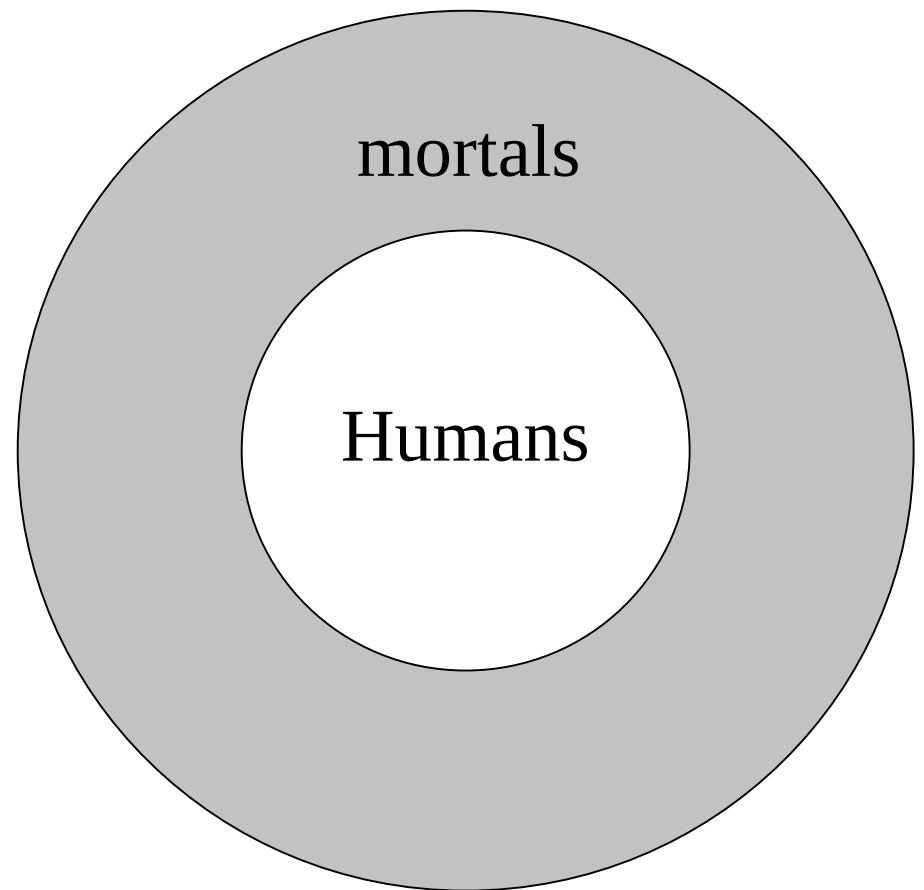
Combine the major and minor premise diagrams:

This is the conclusion.

1) Is there only one way to combine the premise diagrams?

- Zeus

2) Does the combined diagram match the expected conclusion?



Practice Problems

- Use modus ponens or modus tollens to fill in the blank
 - If you do not freeze, then I will shoot.
 - You did not freeze.
 - Therefore: I shot.
- Uses Modus Ponens.

Practice Problems

- Use modus ponens or modus tollens to fill in the blank
 - If they were unsure of the address, then they would have telephoned.
 - ~~They did not telephone.~~
 - Therefore, they were sure of the address.
- Uses Modus Tollens.

Practice Problems

- Use modus ponens or modus tollens to fill in the blank
 - If the moon is made of cheese, it is Wednesday.
 - The moon is made of cheese.
 - Therefore: It is Wednesday.
- Uses Modus Ponens.

Practice Problems

- Use modus ponens or modus tollens to fill in the blank
 - If $\sqrt{2}$ is rational, then $\sqrt{2} = a/b$ for some integers a and b .
 - It is not true that $\sqrt{2} = a/b$ for some integers a and b .
 - Therefore, $\sqrt{2}$ is not rational.
- Uses Modus Tollens.

Definition: A conjunction is another way of saying “and” or “ $\wedge$.”

Definition: Disjunction is another way of saying “or” or “ $\vee$.”

Disjunctive Addition

(method of generalizing)

Arbitrary Form

Instantiated Form

p

He is being tickled.

$\therefore p \vee q$

$\therefore$ He is being tickled or he is sad

q

He is hungry.

$\therefore p \vee q$

$\therefore$ He is hungry or he is Swiss

Conjunctive Addition

(Formalization of the definition)

Arbitrary Form

Instantiated Form

p

The dog is smelly.

q

The dog has no nose.

$\therefore p \wedge q$

$\therefore$ The dog has no nose and smells.

Conjunctive Simplification

(method of particularization)

Arbitrary Form

Instantiated Form

$p \wedge q$

He is sad and he is eating.

$\therefore p$

$\therefore$ He is sad.

$p \wedge q$

He is hungry and he is Swiss.

$\therefore q$

$\therefore$ He is Swiss.

Disjunctive Syllogism

(method of “ruling-out”)

Arbitrary Form

Instantiated Form

$p \vee q$

He is sad or he is eating.

$\sim q$

He is not sad.

$\therefore p$

$\therefore$ He is eating.

$p \vee q$

He is hungry or he is Swiss.

$\sim p$

He is not Swiss

$\therefore q$

$\therefore$ He is hungry.

Valid Inference Rules: Disjunctive Syllogism

$p \vee q \vee r$; It's red, blue, or green.

$\sim r$; It's not green

$\therefore p \vee q$ $\therefore$ It's red or blue.

p	q	r	$p \vee q \vee r$	$\sim r$	$p \vee q$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	T	F	T
T	F	F	T	T	T
F	T	T	T	F	T
F	T	F	T	T	T
F	F	T	T	F	F
F	F	F	F	T	F

Hypothetical Syllogism

(transitivity of implication ($\Rightarrow$))

Arbitrary Form

Instantiated Form

$$p \Rightarrow q$$

If Henry is teething he will cry.

$$q \Rightarrow r$$

If Henry is crying he will not sleep.

$$\therefore p \Rightarrow r$$

$\therefore$ If Henry is teething he will not sleep.

Dilemma

(Greek, Di: “two”, lemma: “take”)

(Division into cases)

Arbitrary Form

$$p \vee q$$

$$p \Rightarrow r$$

$$q \Rightarrow r$$

$$\therefore r$$

Instantiated Form

x is positive or x is negative.

If x is positive then $x^2 > 0$.

If x is negative then $x^2 > 0$.

$\therefore x^2 > 0$.

Practice Problem

- Use valid inference rules to create new premises that imply the conclusion.

A: $\sim p \vee q \Rightarrow r$

B: $s \vee \sim q$

C: $\sim w$

D: $p \Rightarrow w$

E: $\sim p \wedge r \Rightarrow \sim s$

F: $D \wedge C \Rightarrow \sim p$ (Modus Tollens)

G: $F \Rightarrow \sim p \vee q$ (Disjunctive Addition)

H: $F \wedge A \Rightarrow r$ (Modus Ponens)

I: $F \wedge H \Rightarrow \sim p \wedge r$ (Conjunctive Addition)

J: $I \wedge E \Rightarrow \sim s$ (Modus Ponens)

K: $J \wedge B \Rightarrow \sim q$ (Disjunctive Syllogism)

- Conclusion:

- Therefore, $\sim q$

Where are my glasses?

- A: If my glasses are on the kitchen table then I saw them at breakfast.
- B: I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.
- C: If I was reading the newspaper in the living room then my glasses are on the coffee table.
- D: I did not see my glasses at breakfast.
- E: If I was reading my book in bed then my glasses are on the the bedside table.
- G: If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

Solution

If my glasses are on the kitchen table then I saw them at breakfast.

I did not see my glasses at breakfast.

∴ My glasses are not on the kitchen table.

If I was reading the paper in the kitchen, then my glasses are on the kitchen table.

My glasses are not on the kitchen table.

∴ I did not read the paper in the kitchen.

I was reading the paper in the living room or I was reading it in the kitchen.

I did not read the paper in the kitchen.

∴ I was reading the paper in the living room

If I was reading the paper in the living room then my glasses are on the coffee table.

I was reading the paper in the living room.

∴ My glasses are on the coffee table.

Solution Symbolically -- The Form

- Let p be “my glasses are on the kitchen table”.
- Let q be “I saw my glasses at breakfast”.
- Let r be “I was reading the newspaper in the living room”.
- Let s be “I was reading the newspaper in the kitchen”.
- Let w be “my glasses are on the coffee table”.
- Let u be “I was reading my book in bed”.
- Let v be “my glasses are on the bed table”.

- A: If my glasses are on the kitchen table then I saw them at breakfast. $p \Rightarrow q$
- B: I was reading the newspaper in the living room or I was reading the newspaper in the kitchen. $r \vee s$
- C: If I was reading the newspaper in the living room then my glasses are on the coffee table. $r \Rightarrow w$
- D: I did not see my glasses at breakfast. $\sim q$
- E: If I was reading my book in bed then my glasses are on the the bedside table. $u \Rightarrow v$
- G: If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table. $s \Rightarrow p$

Using the formal representation we can deduce w:

$$p \Rightarrow q$$

$$r \vee s$$

$$r \Rightarrow w$$

$$\sim q$$

$$u \Rightarrow v$$

$$s \Rightarrow p$$

$$p \Rightarrow q$$

$$\sim q$$

$$\therefore \sim p$$

$$r \vee s$$

$$\sim s$$

$$\therefore r$$

$$s \Rightarrow p$$

$$\sim p$$

$$\therefore \sim s$$

$$r \Rightarrow w$$

$$r$$

$$\therefore w$$

Rule Of Contradiction

- Definition:
 - If you can show that a supposition “the statement p is false” leads to a contradiction, then you can conclude the statement p is true.
- Formally:
$$\sim p \Rightarrow c \quad (\text{where } c \text{ is a contradiction})$$
$$\therefore p$$

Knights and Knaves

An island is inhabited by knights and knaves.

Knaves always lie.

Knights always tell the truth.

You have met two inhabitants (A and B) of the island:

A says: B is a knight.

B says: A is not the same as me.

What are A and B?

Suppose A is a Knight.

- ∴ What A says is true. (By definition of Knight)
- ∴ B is also a knight. (A said so and A tells the truth)
- ∴ B tells the truth. (By definition of knight)
- ∴ A and B are of opposite types. (B said so and B tells the truth)
- ... but this is a contradiction!

If A is a knight (the supposition) then it logically follows that B is also a knight. And it also follows that B is not the same type as A.

- ∴ The supposition is false. (By rule of contradiction)
- ∴ A is a knave. (By disjunctive syllogism)
- ∴ B is not a knight. (Since we know know A lies)
- ∴ B is a knave (By disjunctive syllogism)
- ∴ A and B are both knaves.

Valid inferences with false conclusions:

- Example:
 - If John Lennon was a rock star, then John Lennon had red hair.
 - John Lennon was a rock star.
 - Therefore, John Lennon had red hair.
- The conclusion is false because the premise is an incorrect statement.
- The inference is still valid.

Invalid Argument With A True Conclusion

- Example:
 - If New York is a big city, then New York has tall buildings.
 - New York has tall buildings.
 - Therefore, New York is a big city.
- The conclusion is a correct statement.
- The way we got it uses an invalid argument based on the common mistake called **converse error**.

Proofs

What is a Proof?

- A proof is a formal argument for the truth of some statement.
- A proof is an algorithm for demonstrating the truth of a statement and as such is like writing a computer program.
- A proof is a sequence of premises derived from previous premises using valid inference rules.
- The last premise in the proof is the conclusion and is what was to be proven.

Anatomy of a proof:

State the proposition to be proven as formally as possible.

Proof:

Inference (*Axiom, Inference Rule, or Definition used*)

...

Inference (*Axiom, Inference Rule, or Definition used*)

Q.E.D or $\square$

Direct Proof

$\forall x \in \text{Domain } D, \text{ if } P(x) \text{ then } Q(x)$

Suppose that $x \in D$, and that $P(x)$ is true.

We would like to show that $Q(x)$ can be shown from $P(x)$.

We can use:

- Definitions

- Previous Proofs

- Valid Inference Rules

Direct Proof 1

- If the sum of two integers is even then so is the difference of those two integers.

Formally: $\forall x, y \in \mathbf{Z}, \text{even}(x + y) \Rightarrow \text{even}(x - y)$

Background Research:

Definition of even:

n is even if $\exists k \in \mathbf{Z} \ni n = 2k$

Definition of odd:

n is odd if $\exists k \in \mathbf{Z} \ni n = 2k + 1$

Direct Proof 1

- If the sum of two integers is even then so is the difference of those two integers.

Formally: $\forall x, y \in \mathbf{Z}, \text{even}(x + y) \Rightarrow \text{even}(x - y)$

Proof:

Let m and n be integers, such that $m + n$ is even.

$m + n = 2k$ for some integer k (By definition of even)

$m = 2k - n$ (Subtract n from both sides, algebra)

$m - n = 2k - 2n$ (Subtract n from both sides, again)

$m - n = 2(k - n)$ (Factor out the two, arithmetic)

...but $k - n$ is just some integer j (Integers closed under subtraction)

$m - n = 2j$ (Substitute j for $k - n$, algebra)

By definition of even $m - n$ is even since it has the form $2j$.

Q. E. D.

Direct Proof 2

- The sum of any two rational numbers is rational.
(Closure of rational number under addition)

Formally: $\forall x, y \in \mathbf{Q}, (x + y) \in \mathbf{Q}$

Background Research:

Definition of rational number:

A rational number can be written as the quotient of two integers.

Formally: $\forall x \in \mathbf{Q}, \exists a, b \in \mathbf{Z} \ni x = a/b$

Direct Proof 2

Formally: $\forall x, y \in \mathbf{R}, (x + y) \in \mathbf{R}$

Proof:

Let m and n be rational numbers.

$$m = \frac{a}{b} \quad n = \frac{c}{d} \quad (\text{Definition of Rational})$$

$$m + n = \frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd} \quad (\text{Arithmetic})$$

Let the integer $p = ad + bc$ **(Integers are closed over arithmetic)**

Let the integer $q = bd$ **(q is non-zero since b and z are non-zero)**

$$m + n = \frac{p}{q}, \quad p, q \in \mathbf{Z} \wedge q \neq 0 \quad (\text{Substitution})$$

$m + n$ is rational since it is a quotient of integers.

Q. E. D.

Proof by Counterexample

- Good for proving that a universal statement is false.

For example:

Prove that the statement "For all real numbers x and y , if $x^2 = y^2$ then $x = y$ " is false.

Proof by Counterexample

Prove that : If x^2 is equal to y^2 x does not necessarily equal y .

Formally: $\sim \forall x, y \in R, (x^2 = y^2) \Rightarrow (x = y)$

Or : $\exists x, y \in R \ni (x^2 = y^2) \wedge (x \neq y)$

Proof by Counterexample:

Let m be the real number 3

Let n be the real number -3

$$3^2 = (-3)^2 \quad \text{i.e.} \quad 9 = 9$$

$$3 \neq -3$$

Q. E. D.

Indirect Proofs

- What are indirect proofs

Proof by Contradiction

Approach:

- 1) Negate the statement to be proved.
- 2) Derive a contradiction using the negated statement.
- 3) Since the negated statement caused a paradox the negated statement cannot be true.
- 4) If the negated statement is false then the statement itself must be true.

Contradiction Proof 1

Prove that $\sqrt{2}$ is an irrational number

Stated Formally: $\forall a, b \in \mathbb{Z}, \frac{a}{b} \neq \sqrt{2}$

Negate the formal expression: $\exists a, b \in \mathbb{Z} \ni \frac{a}{b} = \sqrt{2}$

Contradiction Proof 1

Prove that $\sqrt{2}$ is irrational.

Formally: $\forall a, b \in \mathbb{Z}, \frac{a}{b} \neq \sqrt{2}$

Proof by contradiction:

Suppose that $\sqrt{2}$ is rational, i.e. $\exists a, b \in \mathbb{Z} \ni \frac{a}{b} = \sqrt{2}$

$\sqrt{2} = \frac{m}{n}$, where m and n are integers and have no

common factors (definition of rational number)

$$2 = \frac{m^2}{n^2} \quad (\text{Square both sides, algebra})$$

$$m^2 = 2n^2 \quad (\text{Multiply both sides by } n^2, \text{ algebra})$$

m^2 is even (Since $2n^2$ is of the form $2k$, where k is an integer)

Contradiction Proof 1

Prove that $\sqrt{2}$ is irrational.

Formally: $\forall a, b \in \mathbb{Z}, \frac{a}{b} \neq \sqrt{2}$

Proof by contradiction:

...

$m^2 = 2n^2$ (Multiply both sides by n^2 , algebra)

m^2 is even (Since $2n^2$ is of the form $2k$, where k is an integer)

Since m^2 is even m must also be even (Lemma)

$m = 2k$ (Definition of even)

$m^2 = (2k)^2 = 4k^2 = 2n^2$ (Substitution)

$2k^2 = n^2$ (Divide $4k^2 = 2n^2$ by 2)

$2j = n$ and is even (The square root of an even number is even)

Contradiction Proof 1

Formally: $\forall a, b \in \mathbb{Z}, \frac{a}{b} \neq \sqrt{2}$

Proof by contradiction:

...

$\sqrt{2} = \frac{m}{n}$, where m and n are integers and have no

common factors (definition of rational number)

...

$2j = n$ and is even (The square root of an even number is even)

We know now that m and n are both even. Since they are both even they share the factor 2. This contradicts our earlier premise.

Since this paradox was logically derived from the supposition that $\sqrt{2}$ is rational we know that our supposition was wrong and that $\sqrt{2}$ is irrational. (By rule of contradiction)

Q.E.D.

Fallacies

- Common mistakes we make:
 - using vague or ambiguous premises
 - assuming what is to be proved
 - jumping to conclusions
 - begging the question
- Two others that look like modus ponens and modus tollens (again)
 - converse error
 - assuming a statement is the **converse** of what is stated
 - inverse error
 - assuming a statement is the **inverse** of what is stated