Problem 1. 10 points

Write each of the following sets by listing their elements (Assume the domain of discourse is the integers):

(a) $A = \{x \in \mathbb{Z} \mid 5 \leq x \leq 8\}$.

(b) $A = \{x \in \mathbb{Z} \mid 6 < x < 10\}$.

(c) $A = \{x \in \mathbb{Z} \mid 6 < x < 10 \wedge 4 < x < 12\}$.

(d) $A = \{x \in \mathbb{Z} \mid 6 < x < 10 \vee 4 < x < 12\}$.

(e) $A = \{x \in \mathbb{Z} \mid 6 < x < 8 \vee 8 < x < 12\}$.

(f) $A = \{x \in \mathbb{Z} \mid 6 < x < 8 \wedge 8 < x < 12\}$.

(g) $A = \{x \in \mathbb{Z} \mid 6 < x < 8\} \cup \{x \in \mathbb{Z} \mid 8 < x < 12\}$.

(h) $A = \{x \in \mathbb{Z} \mid 6 < x < 8\} \cap \{x \in \mathbb{Z} \mid 8 < x < 12\}$.

(i) $A = \{x \in \mathbb{Z} \mid 6 \leq x \leq 8\} \setminus \{x \in \mathbb{Z} \mid 8 \leq x \leq 12\}$.

(j) $A = \overline{\{2, 3, 4, 5\}} \cap \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Solution

(a)

{5, 6, 7, 8}

(b)

{7, 8, 9}

(c)

{7, 8, 9}

(d)

{5, 6, 7, 8, 9, 10, 11}

(e)

{7, 9, 10, 11}

(f)

$\emptyset$

(g)

$\{7, 9, 10, 11\}$

(h)

$\emptyset$

(i)

$\{6, 7\}$

(j)

$\{1, 6, 7, 8, 9, 10\}$

Problem 2. 10 points

For the universal set U = $\mathbb{Z}$ draw a Venn diagram for three sets A, B and C showing the locations of the elements of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ when all the following conditions are met.

(a) $A \cap B \cap C = \{5\}$.

(b) $\overline{A \cup B \cup C} = \{10\}$.

(c) $A \setminus (B \cup C) = \{3\}$.

(d) $B \setminus (A \cup C) = \{4\}$.

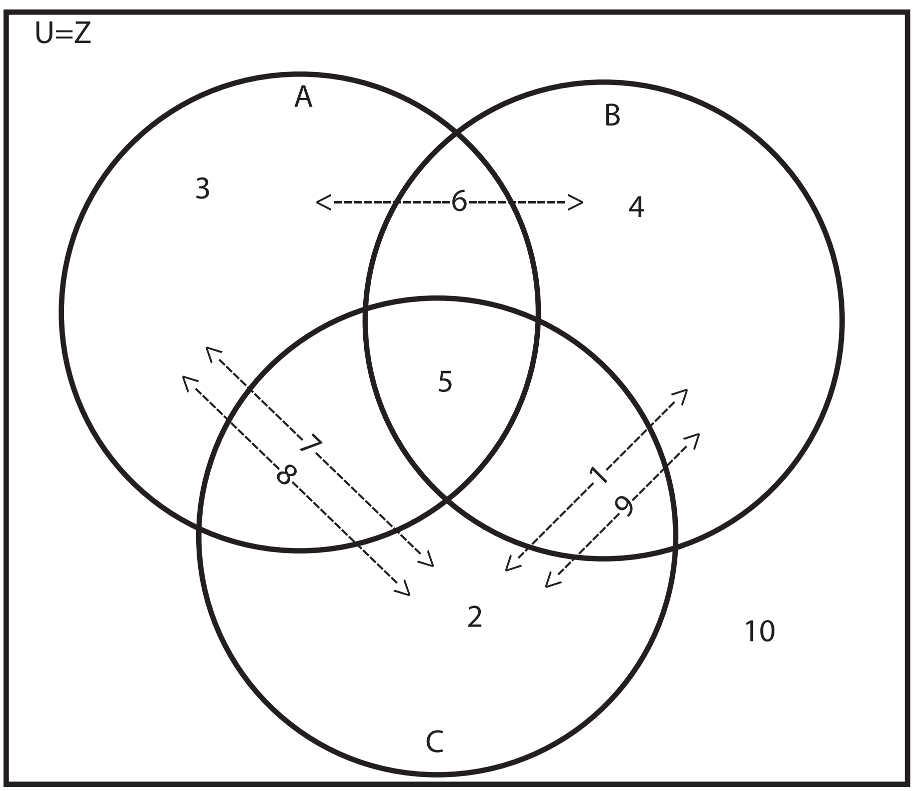
(e) $C \setminus (A \cup B) = \{2\}$.

(f) $(B \cup C) \setminus A = \{1, 2, 4, 9\}$.

(g) $(A \cup C) \setminus B = \{2, 3, 7, 8\}$.

(h) $(A \cup B) \setminus C = \{3, 4, 6\}$.

Solution



Problem 3. 12 points

Let $S = \{a, b, c, d, e, f, g\}$ and $Q = \{1, 2, 3, 4\}$. (Notice $Q \neq \mathbb{Q}$)

(a) Write $\mathcal{P}(Q)$.

(b) Give an example of a set $A \subseteq \mathcal{P}(S)$ such that $|A| = 2$.

(c) Give an example of a set $B \in \mathcal{P}(S)$ such that $|B| = 3$.

(d) Write $C \times D$ where $C = \{x \in Q \mid \text{even}(x)\}$ and $D = \{x \in S \mid \text{vowel}(x)\}$.

(e) Write $A \times B$ where $A = \{1, \{1\}\}$ and $B = \mathcal{P}(A)$.

(f) Write $|\mathcal{P}(A)|$ where $|A| = 512$.

Solution

(a)

$\{\{1,2,3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2\}, \{1,3\},$
$\{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1\}, \{2\}, \{3\}, \{4\}, \emptyset\}$

(b)

$\{1, 2\}$

(c)

$\{1, 2, 3\}$

(d)

$\{(2, a), (2, e), (4, a), (4, e)\}$

(e)

$B = \mathcal{P}(A) = \{\{1, \{1\}\}, \{1\}, \{\{1\}\}, \emptyset\}$, $A \times B = \{(1, \{1, \{1\}\}), (1, \{1\}), (1, \{\{1\}\}), (1, \emptyset),$
$(\{1\}, \{1, \{1\}\}), (\{1\}, \{1\}), (\{1\}, \{\{1\}\}), (\{1\}, \emptyset)\}$

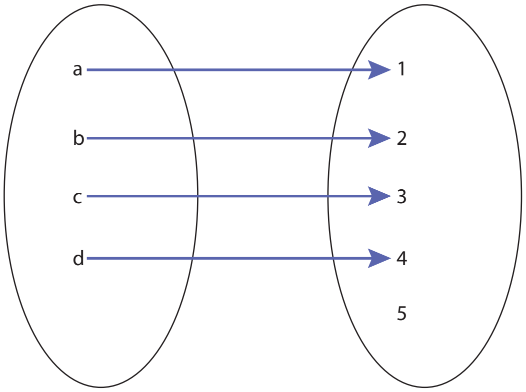
(f)

13,407,807,929,942,597,099,574,024,998,205,846,127,479,365,820,592,393,377,723,561,443,721,764,
030,073,546,976,801,874,298,166,903,427,690,031,858,186,486,050,853,753,882,811,946,569,946,
433,649,006,084,096

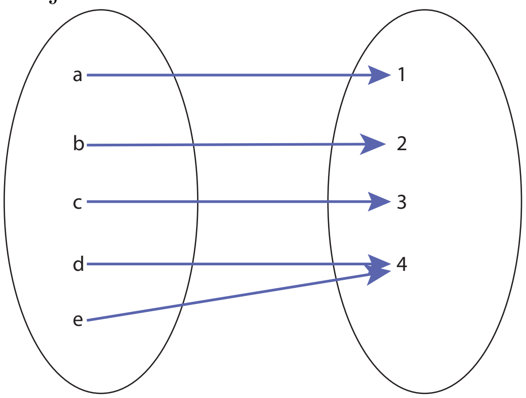
Problem 4. 9 points

Draw arrows from the domain A to the codomain B where $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4, 5\}$ such that (a) is an injective function, (b) is a sujective function and (c) is a bijective function. You may need to erase some values from the domain and codomain to avoid having all the solutions be trivially bijective.

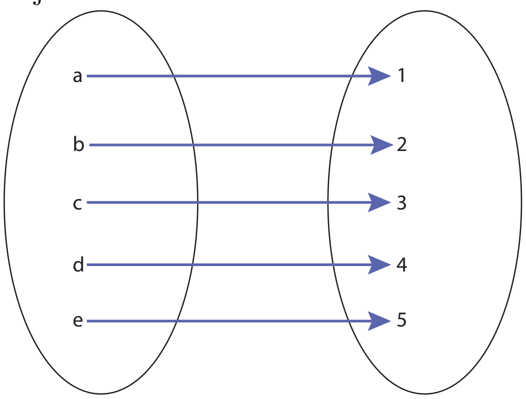
(a) Injective



(b) Surjective



(c) Bijective



Problem 5. 20 points

Prove that $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 2x + 1$ is bijective by:

(a) Proving that $f(x) = 2x + 1$ is injective by showing that $f(x) = f(y) \Rightarrow x = y$.

(b) Proving that $f(x) = 2x + 1$ is surjective by showing that $y = f(x) \Rightarrow x \in \mathbb{R}$.

Solution

(a)

Claim: $f(x) = f(y) \Rightarrow x = y$.

Proof.

$f(x) = 2x + 1$	Premise
$f(y) = 2y + 1$	Substitution
$f(x) = f(y)$	Premise
$\implies 2x + 1 = 2y + 1$	Substitution
$\implies 2x = 2y$	Algebra: subtract 1 from both sides
$\implies x = y$	Algebra: divide both sides by 2

□

(b)

Claim: $y = f(x) \Rightarrow x \in \mathbb{R}$.

Proof.

$f(x) = 2x + 1$	Premise
$y = 2x + 1$	Premise and Substitution
$\implies y - 1 = 2x \in \mathbb{R}$	Algebra: subtract 1 from both sides; reals are closed under subtraction
$\implies \frac{y-1}{2} = x \in \mathbb{R}$	Algebra: divide both sides by 2; reals closed under division
$\implies \frac{y-1}{2} \in \mathbb{R} \implies x \in \mathbb{R}$	

□

Problem 6. 16 points

Write out a_0, a_1, a_2, and a_3 for the sequence (a_n):

(a) $(a_n) = (-2)^n$

(b) $(a_n) = 3$

(c) $(a_n) = 7 + 4^n$

(d) $(a_n) = 2^n + (-2)^n$

Solution

(a)

1, -2, 4, -8

(b)

3, 3, 3, 3

(c)

8, 11, 23, 71

(d)

2, 0, 8, 0

Write the values of the following summations:

(e) $\sum_{j=0}^{8} (1 + (-1)^j)$

(f) $\sum_{j=0}^{8} (3^j - 2^j)$

(g) $\sum_{j=0}^{8} (2 \cdot 3^j + 3 \cdot 2^j)$

(h) $\sum_{j=0}^{8} (2^{j+1} - 2^j)$

Solution

(e)

$= (1 + (-1)^0) + (1 + (-1)^1) + (1 + (-1)^2) + (1 + (-1)^3) + (1 + (-1)^4) + (1 + (-1)^5) + (1 + (-1)^6) + (1 + (-1)^7) + (1 + (-1)^8)$
$= 2 + 0 + 2 + 0 + 2 + 0 + 2 + 0 + 2 = 10$

(f)

$(3^0-2^0)+(3^1-2^1)+(3^2-2^2)+(3^3-2^3)+(3^4-2^4)+(3^5-2^5)+(3^6-2^6)+(3^7-2^7)+(3^8-2^8) = 9841 - 511 = 9330$

(g)

$\sum_{j=0}^{8}(2 \cdot 3^j + 3 \cdot 2^j) = 2\sum_{j=0}^{8} 3^j + 3\sum_{j=0}^{8} 2^j = 2 \cdot 9841 + 3 \cdot 511 = 19682 + 1533 = 21215$

(h)

$\sum_{j=0}^{8}(2^{j+1} - 2^j) = \sum_{j=0}^{8} 2^{j+1} - \sum_{j=0}^{8} 2^j$

All the terms cancel except the first in the second sum and the last in the first sum:
$-2^0 + 2^{8+1} = -1 + 512 = 511$

Problem 7. 8 points

Write the values of the following summations:

(a) $\sum_{i=1}^{3}\sum_{j=1}^{2}(i-j)$

(b) $\sum_{i=0}^{3}\sum_{j=0}^{2}(3i+2j)$

(c) $\sum_{i=1}^{3}\sum_{j=0}^{2}(j)$

(d) $\sum_{i=0}^{2}\sum_{j=0}^{3}(i^2j^3)$

Solution

(a)

$$\begin{aligned}\sum_{i=1}^{3}\sum_{j=1}^{2}(i-j) &= (1-1)+(1-2)+(2-1)+(2-2)+(3-1)+(3-2)\\ &= 0-1+1+0+2+1\\ &= 3\end{aligned}$$

(b)

$$\begin{aligned}\sum_{i=0}^{3}\sum_{j=0}^{2}(3i+2j) &= [(3\cdot 0+2\cdot 0)+(3\cdot 0+2\cdot 1)+(3\cdot 0+2\cdot 2)]\\ &+[(3\cdot 1+2\cdot 0)+(3\cdot 1+2\cdot 1)+(3\cdot 1+2\cdot 2)]\\ &+[(3\cdot 2+2\cdot 0)+(3\cdot 2+2\cdot 1)+(3\cdot 2+2\cdot 2)]\\ &+[(3\cdot 3+2\cdot 0)+(3\cdot 3+2\cdot 1)+(3\cdot 3+2\cdot 2)]\\ &= 0+2+4\\ &+3+5+7\\ &+6+8+10\\ &+9+11+13\\ &= 78\end{aligned}$$

(c)

$$\sum_{i=1}^{3}\sum_{j=0}^{2}(j) = [0+1+2]+[0+1+2]+[0+1+2] = 3+3+3 = 9$$

(d)

$$\begin{aligned}\sum_{i=0}^{2}\sum_{j=0}^{3}\left(i^2j^3\right) &= [(0^20^3)+(0^21^3)+(0^22^3)+(0^23^3)] \\ &+ [(1^20^3)+(1^21^3)+(1^22^3)+(1^23^3)] \\ &+ [(2^20^3)+(2^21^3)+(2^22^3)+(2^23^3)] \\ &= 0+0+0+0 \\ &+ 0+1+8+27 \\ &+ 0+4+4\cdot 8+4\cdot 27 \\ &= 1+8+27+4+32+108 \\ &= 180\end{aligned}$$

Problem 8. 15 points

Use the *formulas* for arithmetic series to answer (a) and (b).

(a) Find the 15th term of the arithmetic sequence 3, 5, 7, 9, ...

(b) Find the sum of the first 20 terms in the sequence 3, 5, 7, 9, ...

(c) Find the value of the geometric series: $-2, \frac{1}{2}, -\frac{1}{8}, ..., -\frac{1}{32768}$

Solution

(a)

$a_0 = 3, d = a_1 - a_0 = 2, n + 1 = 15$ so $n = 14$
The fifteenth term is $a_0 + nd = 3 + 14 \cdot 2 = 31$

(b)

The twentieth term is $a_0 + nd = 3 + 2 \cdot 19 \cdot 2 = 41$
$n = 20$ so $\frac{na_0 + na_{n-1}}{2} = \frac{20 \cdot 3 + 41 \cdot 20}{2}$
$= \frac{60 + 41 \cdot 20}{2} = 30 + 41 \cdot 10 = 30 + 410 = 440.$

(c)

This is a geometric series so we need to find $a_0, r,$ and n and use the formula $\frac{a_0(1-r^n)}{1-r}$ to find the sum.
$a_0 = -2$
$r = \frac{a_0}{a_1} = \frac{\frac{1}{2}}{-2} = \frac{1}{2} \cdot \frac{1}{-2} = -\frac{1}{4}.$

The last term in the sequence is $-\frac{1}{32768}$. We can equate this term with the formula for the term with index m and solve for m.
$-\frac{1}{32768} = a_0 r^m = -2 \cdot \left(-\frac{1}{4}\right)^m$

Solving for m:
$-\frac{1}{32768} = -2 \cdot \left(-\frac{1}{4}\right)^m$

$\frac{1}{65536} = \left(-\frac{1}{4}\right)^m$

$\frac{1}{65536} = \frac{1}{(-4)^m}$

Need to figure out what power to raise -4 by to get 65536.

Powers of $-4 = -4, 16, -64, 256, -1024, 4096, -16384, 65536$. Bingo! $m = 8$.
If the index of the last element is 8 there must be $n = 9$ terms in the sequence since we started with index 0 as the first element.

Now we can plug the values into the formula: $\frac{a_0(1-r^n)}{1-r} = \frac{-2\left(1-\left(-\frac{1}{4}\right)^9\right)}{1-\left(-\frac{1}{4}\right)} = -\frac{262143}{163840}$.

Problem 9. 10 points

Mark the congruent expressions:

(a) $24 \stackrel{?}{\equiv} 14 (\text{mod } 5)$ ✓

(b) $-7 \stackrel{?}{\equiv} 9 (\text{mod } 8)$ ✓

(c) $12 \stackrel{?}{\equiv} 12 (\text{mod } 9)$ ✓

(d) $34 \stackrel{?}{\equiv} 16 (\text{mod } 12)$

(e) $24 \stackrel{?}{\equiv} 3 (\text{mod } 7)$ ✓

(f) $-17 \stackrel{?}{\equiv} 9 (\text{mod } 8)$

(g) $-5 \stackrel{?}{\equiv} -5 (\text{mod } 4)$ ✓

(h) $24 \stackrel{?}{\equiv} -3 (\text{mod } 3)$ ✓

(i) $47 \stackrel{?}{\equiv} 23 (\text{mod } 8)$ ✓

(j) $18 \stackrel{?}{\equiv} 38 (\text{mod } 5)$ ✓

Problem 10. 8 points

Use the Sieve of Eratosthenes to find all the prime numbers less than 30.

Solution

Only need to consider prime factors up to $\lceil\sqrt{30}\rceil = 6$.
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29

Eliminate multiples of 2:

1, ~~2~~, 3, ~~4~~, 5, ~~6~~, 7, ~~8~~, 9, ~~10~~, 11, ~~12~~, 13, ~~14~~, 15, ~~16~~, 17, ~~18~~, 19, ~~20~~, 21, ~~22~~, 23, ~~24~~, 25, ~~26~~, 27, ~~28~~, 29

Eliminate multiples of 3:

1, 2, 3, ~~4~~, 5, ~~6~~, 7, ~~8~~, ~~9~~, ~~10~~, 11, ~~12~~, 13, ~~14~~, ~~15~~, ~~16~~, 17, ~~18~~, 19, ~~20~~, ~~21~~, ~~22~~, 23, ~~24~~, 25, ~~26~~, ~~27~~, ~~28~~, 29

Eliminate multiples of 5:

1, 2, 3, ~~4~~, 5, ~~6~~, 7, ~~8~~, ~~9~~, ~~10~~, 11, ~~12~~, 13, ~~14~~, ~~15~~, ~~16~~, 17, ~~18~~, 19, ~~20~~, ~~21~~, ~~22~~, 23, ~~24~~, ~~25~~, ~~26~~, ~~27~~, ~~28~~, 29

The remaining numbers (except 1) are prime: 2, 3, 5, 7, 11, 13, 17, 19, 23 and 29.

Problem 11. 10 points

Use the Euclidean algorithm to calculate the greatest common divisor: gcd(a,b).

(a) $a = 558, b = 26$

(b) $a = 11, b = 19$

(c) $a = 0, b = -10$

(d) $a = -8, b = 14$

(e) $a = 17, b = 51$

Solution

(a)

$\frac{558}{26} = 21$ r $12, \frac{26}{12} = 2$ r $2, \frac{12}{2} = 6$ r $0, \text{gcd}(558, 26) = 2$

(b)

$\frac{19}{11} = 1$ r $8, \frac{11}{8} = 1$ r $3, \frac{8}{3} = 2$ r $2, \frac{3}{2} = 1$ r $1, \frac{2}{1} = 2$ r $0, \text{gcd}(19, 11) = 1$
Or, since 19 is prime its gcd with all integers less than 19 is 1. 11 is less than 19 so their gdc is 1.

(c)

This is a bit of a trick question.
All numbers divide zero since $\forall x \in \mathbb{N}, \frac{0}{x} = 0 \wedge 0 \in \mathbb{Z}$. The largest number that divides -10 is 10 and 10 also divides 0 so 10 is the greatest common divisor.
$\text{gcd}(0, q) = q$ is the base case for the Euclidean algorithm.

(d)

$\text{gcd}(-8, 14):$
$14 = -1 \times -8 + 6$
$-8 = -2 \times 6 + 4$
$6 = 1 \times 4 + 2$
$4 = 2 \times 2 + 0$
$\text{gcd}(-8, 14) = 2$

(e)

$\text{gcd}(17, 51):$
$51 = 3 \times 17 + 0$
$\text{gcd}(17, 51) = 17$

Problem 12. 10 points

Calculate the following where $\Phi(x)$ is Euler's Totient function.

(a) $\Phi(3)$

(b) $\Phi(18)$

(c) $\Phi(27)$

(d) $\Phi(15)$

(e) $\Phi(997)$

Solution

(a)

3 coprime with 1 and 2 so $\Phi(3) = 2$.

(b)

18 coprime with 1, 5, 7, 11, 13, 17 so $\Phi(18) = 6$.

(c)

27 coprime with 1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23, 25, and 26 so $\Phi(27) = 18$.

(d)

15 coprime with 1, 2, 4, 7, 8, 11, 13, 14 so $\Phi(15) = 8$.

(e)

997 is prime so the totient is 996.