Problem 1. 10 points

Write each of the following sets by listing their elements (Assume the domain of discourse is the integers):

(a) $A = \{x \in \mathbb{Z} \mid 5 \leq x \leq 8\}$.

(b) $A = \{x \in \mathbb{Z} \mid 6 < x < 10\}$.

(c) $A = \{x \in \mathbb{Z} \mid 6 < x < 10 \wedge 4 < x < 12\}$.

(d) $A = \{x \in \mathbb{Z} \mid 6 < x < 10 \vee 4 < x < 12\}$.

(e) $A = \{x \in \mathbb{Z} \mid 6 < x < 8 \vee 8 < x < 12\}$.

(f) $A = \{x \in \mathbb{Z} \mid 6 < x < 8 \wedge 8 < x < 12\}$.

(g) $A = \{x \in \mathbb{Z} \mid 6 < x < 8\} \cup \{x \in \mathbb{Z} \mid 8 < x < 12\}$.

(h) $A = \{x \in \mathbb{Z} \mid 6 < x < 8\} \cap \{x \in \mathbb{Z} \mid 8 < x < 12\}$.

(i) $A = \{x \in \mathbb{Z} \mid 6 \leq x \leq 8\} \setminus \{x \in \mathbb{Z} \mid 8 \leq x \leq 12\}$.

(j) $A = \overline{\{2, 3, 4, 5\}} \cap \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Solution

(a)

(b)

(c)

(d)

(e)

(f)

(g)

(h)

(i)

(j)

Problem 2. 10 points

For the universal set U = $\mathbb{Z}$ draw a Venn diagram for three sets A, B and C showing the locations of the elements of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ when all the following conditions are met.

(a) $A \cap B \cap C = \{5\}$.

(b) $\overline{A \cup B \cup C} = \{10\}$.

(c) $A \setminus (B \cup C) = \{3\}$.

(d) $B \setminus (A \cup C) = \{4\}$.

(e) $C \setminus (A \cup B) = \{2\}$.

(f) $(B \cup C) \setminus A = \{1, 2, 4, 9\}$.

(g) $(A \cup C) \setminus B = \{2, 3, 7, 8\}$.

(h) $(A \cup B) \setminus C = \{3, 4, 6\}$.

Solution

Problem 3. 12 points

Let $S = \{a, b, c, d, e, f, g\}$ and $Q = \{1, 2, 3, 4\}$. (Notice $Q \neq \mathbb{Q}$)

(a) Write $\mathcal{P}(Q)$.

(b) Give an example of a set $A \subseteq \mathcal{P}(S)$ such that $|\{A\}| = 2$.

(c) Give an example of a set $B \in \mathcal{P}(S)$ such that $|B| = 3$.

(d) Write $C \times D$ where $C = \{x \in Q \mid \text{even}(x)\}$ and $D = \{x \in S \mid \text{vowel}(x)\}$.

(e) Write $A \times B$ where $A = \{1, \{1\}\}$ and $B = \mathcal{P}(A)$.

(f) Write $|\mathcal{P}(A)|$ where $|A| = 512$.

Solution

(a)

(b)

(c)

(d)

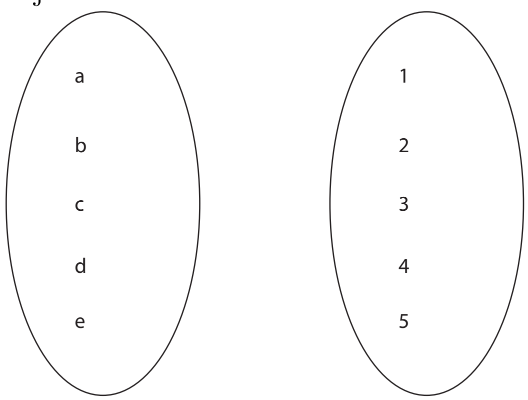
(e)

(f)

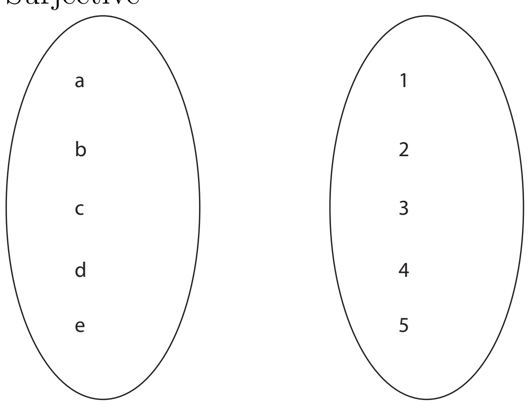
Problem 4. 9 points

Draw arrows from the domain A to the codomain B where $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4, 5\}$ such that (a) is an injective function, (b) is a sujective function and (c) is a bijective function. You may need to erase some values from the domain and codomain to avoid having all the solutions be trivially bijective.

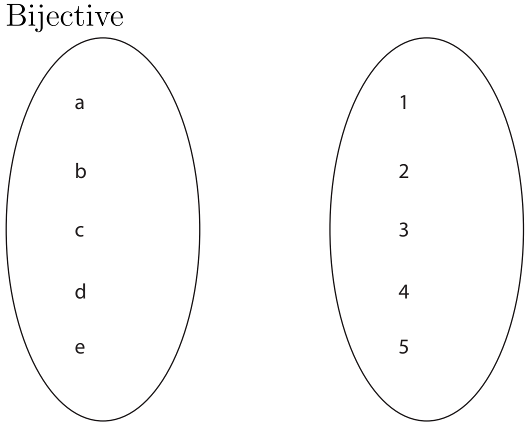
(a) Injective



(b) Surjective



(c) Bijective



Problem 5. 20 points

Prove that $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 2x + 1$ is bijective by:

(a) Proving that $f(x) = 2x + 1$ is injective by showing that $f(x) = f(y) \Rightarrow x = y$.

(b) Proving that $f(x) = 2x + 1$ is surjective by showing that $y = f(x) \Rightarrow x \in \mathbb{R}$.

Solution

(a)

(b)

Problem 6. 16 points

Write out a_0, a_1, a_2, and a_3 for the sequence (a_n):

(a) $(a_n) = (-2)^n$

(b) $(a_n) = 3$

(c) $(a_n) = 7 + 4^n$

(d) $(a_n) = 2^n + (-2)^n$

Solution

(a)

(b)

(c)

(d)

Write the values of the following summations:

(e) $\sum_{j=0}^{8} (1 + (-1)^j)$

(f) $\sum_{j=0}^{8} (3^j - 2^j)$

(g) $\sum_{j=0}^{8} (2 \cdot 3^j + 3 \cdot 2^j)$

(h) $\sum_{j=0}^{8} (2^{j+1} - 2^j)$

Solution

(e)

(f)

(g)

(h)

Problem 7. 8 points

Write the values of the following summations:

(a) $\sum_{i=1}^{3} \sum_{j=1}^{2} (i - j)$

(b) $\sum_{i=0}^{3} \sum_{j=0}^{2} (3i + 2j)$

(c) $\sum_{i=1}^{3} \sum_{j=0}^{2} (j)$

(d) $\sum_{i=0}^{2} \sum_{j=0}^{3} (i^2 j^3)$

Solution

(a)

(b)

(c)

(d)

Problem 8. 15 points

Use the *formulas* for arithmetic series to answer (a) and (b).

(a) Find the 15th term of the arithmetic sequence 3, 5, 7, 9, ...

(b) Find the sum of the first 20 terms in the sequence 3, 5, 7, 9, ...

(c) Find the value of the geometric series: $-2, \frac{1}{2}, -\frac{1}{8}, ..., -\frac{1}{32768}$

Solution

(a)

(b)

(c)

Problem 9. 10 points

Mark the congruent expressions:

(a) $24 \stackrel{?}{\equiv} 14 (\text{mod } 5)$

(b) $-7 \stackrel{?}{\equiv} 9 (\text{mod } 8)$

(c) $12 \stackrel{?}{\equiv} 12 (\text{mod } 9)$

(d) $34 \stackrel{?}{\equiv} 16 (\text{mod } 12)$

(e) $24 \stackrel{?}{\equiv} 3 (\text{mod } 7)$

(f) $-17 \stackrel{?}{\equiv} 9 (\text{mod } 8)$

(g) $-5 \stackrel{?}{\equiv} -5 (\text{mod } 4)$

(h) $24 \stackrel{?}{\equiv} -3 (\text{mod } 3)$

(i) $47 \stackrel{?}{\equiv} 23 (\text{mod } 8)$

(j) $18 \stackrel{?}{\equiv} 38 (\text{mod } 5)$

Problem 10. 8 points

Use the Sieve of Eratosthenes to find all the prime numbers less than 30.

Solution

Problem 11. 10 points

Use the Euclidean algorithm to calculate the greatest common divisor: gcd(a,b).

(a) $a = 558, b = 26$

(b) $a = 11, b = 19$

(c) $a = 0, b = -10$

(d) $a = -8, b = 14$

(e) $a = 17, b = 51$

Solution

(a)

(b)

(c)

(d)

(e)

Problem 12. 10 points

Calculate the following where $\Phi(x)$ is Euler's Totient function.

(a) $\Phi(3)$

(b) $\Phi(18)$

(c) $\Phi(27)$

(d) $\Phi(15)$

(e) $\Phi(997)$

Solution

(a)

(b)

(c)

(d)

(e)