Each problem is worth 10 points.

1 Propositional Logic

1.1

Mark the propositions in the following list:

(a) Where are you?

(b) Look out for that car!

(c) North Carolina is in the United States. ✓

(d) Load the packages onto the cart.

(e) Linux is an operating system. ✓

(f) Jupiter is the closest planet to the Sun. ✓

(g) if $x = 3$ then $x^2 = 9$. ✓

(h) if $x = 3$ then $x^2 = 6$. ✓

(i) Jane drives a Ford or Bob walks. ✓

(j) $x + 5 = 0$.

1.2

Let p and q be the propositions:
$p =$ I will study hard.
$q =$ I will learn a lot.
Translate the following English sentences into symbols using p and q and logical operators including negations.

(a) I will study hard and I will learn a lot.

Solution

$p \wedge q$

(b) I will study hard or I will learn a lot.

Solution

$p \vee q$

(c) I will not study hard and I will learn a lot.

Solution

$\neg p \wedge q$

(d) I will not study hard or I will learn a lot.

Solution

$\neg p \vee q$

(e) If I will study hard then I will learn a lot.

Solution

$p \implies q$

(f) If I will not study hard then I will not learn a lot.

Solution

$\neg p \implies \neg q$

(g) I will not learn a lot therefore I will not study hard.

Solution

$\neg q \implies \neg p$

(h) I will not study hard therefore I will not learn a lot.

Solution

$\neg p \implies \neg q$

(i) I will not study hard or I will not learn a lot.

Solution

$\neg q \vee \neg p$

(j) I will study hard if and only if, I will learn a lot.

Solution

$q \iff p$

1.3

Let p and q be the propositions:
p = Colonel Mustard, in the hall with a dagger.
q = Miss Scarlett in the billiard room with a lead pipe.
r = Professor Plum murdered Mr Boddy in the kitchen with the candlestick.
Translate the following propositions into English.

(a) $p \vee q$

Solution

Col. Mustard is in the hall with the with a dagger or Miss. Scarlett is in the billiard room with a lead pipe.

(b) $q \Rightarrow r$

Solution

Miss. Scarlet being in the billiard room with a lead pipe implies that Prof. Plum murdered Mr. Boddy in the kitchen with the candlestick.

(c) $(p \vee q) \Rightarrow r$

Solution

Col. Mustard is in the hall with the dagger and Miss. Scarlett is in the billiard room with the lead pipe therefore, Prof. Plum murdered Mr. Boddy in the kitchen with the candlestick.

(d) $(p \wedge q) \Leftrightarrow r$

Solution

Col. Mustard is in the hall with the dagger and Miss. Scarlett is in the billiard room with the lead pipe if, and only if, Prof. Plum murdered Mr. Boddy in the kitchen with the candlestick.

(e) $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$

Solution

Col. Mustard is not in the hall with the dagger and Miss. Scarlett is not in the billiard room with the lead pipe, if and only if, Col. Mustard is not in the hall with the dagger and Miss. Scarlett is not in the billiard room with the lead pipe.

1.4

Write the truth table for $(p \wedge q) \wedge \neg (p \wedge q)$

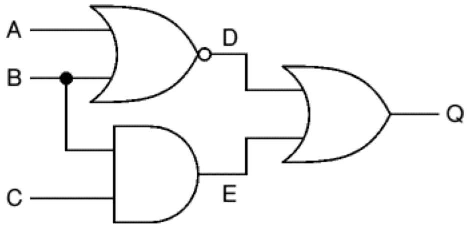
Solution

p	q	$(p \wedge q) \wedge \neg (p \wedge q)$
T	T	F
T	F	F
F	T	F
F	F	F

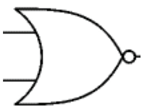
2 Applications

2.1

Give the truth table for this logic circuit and determine the propositional logic formula.



The logic gate below is a nor gate and is equivalent to $\neg(p \vee q)$



Solution

$\neg(A \vee B) \vee (B \wedge C) \equiv Q$

A	B	C	Q
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	1
0	0	0	1

3 Equivalences

3.1

Use truth tables to show whether the following are equivalent propositions:

(a) $(p \vee q) \wedge \neg(p \wedge q)$ and $p \oplus q$

Solution

p	q	$(p \vee q) \wedge \neg(p \wedge q)$	$p \oplus q$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	F	F

The truth values are the same for the two statements so they are equivalent.

(b) $p \wedge (q \wedge r)$ and $(q \vee q) \wedge (q \vee r)$

Solution

p	q	r	$p \wedge (q \wedge r)$	$(q \vee q) \wedge (q \vee r)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	F
T	F	F	F	F
F	T	T	F	T
F	T	F	F	T
F	F	T	F	F
F	F	F	F	F

The truth values are not the same for the two statements so they are not equivalent.

4 Predicate Calculus

4.1

Assign symbols to represent the predicates and write the following in symbolic form:

(a) Some computers are smarter than people.

Solution

$\exists c \in \text{Computers} \ni \exists p \in \text{People} \ni c \text{ smarter than } p.$

(b) For every n, there exist three integers such that the nth power of one is the sum of the nth power of the others.

Solution

$\forall n \in \mathbb{N}, \exists x, y, z \in \mathbb{R} \ni x^n = y^n + z^n.$

(c) He is the best mathematician in the world.

Solution

$\exists h \in \text{Mathematicians}, h \text{ is the best.}$

(d) There are no perfect programs.

Solution

$\forall p \in \text{Programs}, p \text{ is not perfect.}$

(e) For every action there is an equal and opposite reaction.

Solution

$\forall a \in \text{Actions}, \exists r \in \text{Reactions} \ni r \text{ is equal and opposite to } a.$

4.2

Write formal negations of the following statements:

(a) $\forall$ primes p, p is odd .

Solution

$\exists p \ \in \text{Primes}, k \in \mathbb{Z} \ni p = 2k.$

(b) $\exists$ triangle t, sum of the interior angles of t is not equal to $180°$.

Solution

$\forall t \in \text{Triangles}, \text{the interior angles of } p \text{ sum to } 180°.$

(c) $\forall$ politicians p, p is dishonest .

Solution

$\exists p \in \text{Politicians} \ni \text{honest}(p).$

(d) $\exists$ program $p \ni \forall$ programs q, p can tell whether q will terminate..

Solution

$\forall$ program $p, \exists$ programs $q \ni p$ cannot tell whether q will terminate..

(e) All people who are good at football are tall.

Solution

$\exists p \in \text{People} \ni p$ is good at football $\wedge\, p$ is not tall.

4.3

Find counterexamples to the following universal statements:

(a) $\forall x \in \mathbb{R}, x > \frac{1}{x}$.

Solution

$x = 1 \implies 1 > \frac{1}{1} \implies 1 > 1$
$\Rightarrow\Leftarrow$

(b) $\forall a \in \mathbb{Z}, \frac{(a-1)}{a} \notin \mathbb{Z}$.

Solution

$a = 1 \implies \frac{1-1}{1} \notin \mathbb{Z} \implies 0 \notin \mathbb{Z}.$
$\Rightarrow\Leftarrow$

(c) $\forall m, n \in \mathbb{Z}^+, mn \geq m + n$.

Solution

$a = 1, m = 1 \implies 1 \geq 2.$
$\Rightarrow\Leftarrow$

(d) $\forall x, y \in \mathbb{R}, \sqrt{x+y} = \sqrt{x} + \sqrt{y}.$

Solution

$x = 1, y = 1 \implies \sqrt{1+1} = \sqrt{1} + \sqrt{1} \implies 2 = 2\sqrt{2}$
$\Rightarrow\Leftarrow$

(e) $\forall x, y \in \mathbb{R}, x \geq y \wedge z \geq x \Rightarrow z > x.$

Solution

$x = 1, y = 1 \implies 1 \geq 1 \wedge 1 \geq 1$ satisfied but $\implies 1 > 1.$
$\Rightarrow\Leftarrow$

4.4

Let $F(x, y)$ be the statement "x loves y" where the domain is everyone in the world. Use quantifiers to express each of the following:

(a) Everybody loves Tom.

Solution

$\forall x, F(x, \text{Tom}).$

(b) Everybody loves somebody.

Solution

$\forall x, y, F(x, y).$

(c) The is someone that everyone loves.

Solution

$\exists x \ni \forall y, F(y, x).$

(d) There is someone who loves no one besides themselves.

Solution

$\exists x \ni \forall y,\ \text{if } x \neq y \text{ then } \neg F(x,y) \text{ else } F(x,y).$

(e) The is someone who loves everyone.

Solution

$\exists x \ni \forall y, F(x,y).$

4.5

Write a quantified predicate that expresses the distributive law of multiplication over addition for the real numbers.

Solution

$\forall x, y, z \in \mathbb{R}, x(y+z) = xy + xz.$

4.6

Write a quantified predicate that expresses the distributive law of conjunction over disjunction for propositional logic.

Solution

$\forall x, y, z \in \text{Propositions}, x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$

4.7

Write a quantified predicate that expresses the distributive law of conjunction over conjunction for propositional logic.

Solution

$\forall x, y, z \in \text{Propositions}, x \wedge (y \wedge z) = (x \wedge y) \wedge (x \wedge z).$

5 Logical Inference

Name the following argument forms and say whether they are valid or a fallacy. Give an example in plain English that demonstrates the argument form or fallacy.

$a \vee b$
$\neg a$
$\therefore \neg b$

Solution

I am being chased by a bear or a lion. I am not being chased by a lion. Therefore I am not being chased by a bear. Fallacy.

$a \vee b$
$\neg a \vee c$
$\therefore b \vee c$

Solution

Valid.
I am eating an apple or a banana. I am not eating an apple or I am eating a peach. Therefore I am eating a banana or a peach.

a
$\therefore a \vee b$

Solution

Valid.
I am driving a car. Therefore I am driving a car or riding my bike.

$b \wedge (a \to b)$
$\therefore a$

Solution

Invalid
Stan: You gotta spend money to make money.
Francine: But you didn't make any money!
Stan: So logically, I didn't spend any money!

$\neg a \wedge (a \to b)$
$\therefore \neg b$

Solution

Invalid
Wearing a hat implies you have a head. I am not wearing a hat so I must have no head.

$\neg a \wedge (a \rightarrow b)$
$\therefore b$

Solution

Invalid
If the cat is black it rolled in the ink. The cat is not black so it must have rolled in the ink.

$a \rightarrow b$
$\therefore \neg b \rightarrow \neg a$

Solution

Valid
If you have money I gave it to you. I didn't give you money so you don't have any.

$a \rightarrow b$
$\therefore \neg a \rightarrow \neg b$

Solution

] Invalid
If we swam all day we are tired. We did not swim all day so we are not tired.

$a \rightarrow b$
$\therefore b \rightarrow a$

Solution

Invalid
If I worked hard I would have done well. I did well so I must have worked hard.

5.1

Give English examples of argument by Modus ponens,

Solution

If Bernard was born in Switzerland then Bernard is Swiss. Bernard was born in Switzerland so Bernard is Swiss.

Modus tollens,

Solution

If Bernard was born in Switzerland then Bernard is Swiss. Bernard is not Swiss so Bernard was not born in Switzerland.

and Syllogism,

Solution

If a concave shape is a regular polygon with n sides then the sum of the interior angles is $180(n-2)^\circ$. If a regular polygon is a hexagon then it has 6 sides. The interior angles of a hexagon add up to 720°.

5.2

Use rules of inference to show that the following arguments are valid:

Solution

(a) $\neg(s \wedge t)$
$\neg w \rightarrow t$
$\therefore s \rightarrow w$

Solution

$\neg(s \wedge t) \equiv \neg s \vee \neg t$	Premise and De Morgan's Rule
$\neg s \vee \neg t \equiv s \implies \neg t$	Equivalence of $p \implies q$ and $\neg p \vee q$
$\neg w \implies t \equiv \neg t \implies w$	Contrapositive applied to premise
$s \implies w$	Hypothetical Syllogism

(b) $\neg(\neg p \vee q)$
$\neg z \rightarrow \neg s$
$(p \wedge \neg q) \rightarrow s$
$\neg z \vee r$
$\therefore r$

Solution

$p \wedge \neg q$	Premise and De Morgan's Rule
$(p \wedge \neg q) \wedge (p \wedge \neg q) \to s) \implies s$	Modus ponens and premise
$s \implies z$	Premise and contrapositive
$(s \implies z) \wedge s$	Modus ponens
$\neg z \vee r \equiv z \implies r$	Equivalence of $p \implies q$ and $\neg p \vee q$
$z \wedge (z \implies r) \implies r$	Modus ponens

5.3

Use truth tables to show the following inference rules are valid:

(a) $p \to q$
$q \to r$
$\therefore p \to r$

Solution

p	q	r	$p \to q$	$q \to r$	$p \to r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

The critical rows are highlighted. Everywhere that the premises are true the conclusion is true, therefore the argument form is valid.

(b) $p \to q$
$\neg r \to \neg q$
$\neg r$
$\therefore \neg p$

Solution

p	q	r	$p \to q$	$\neg r \to \neg q$	$\neg r$	$\neg p$
T	T	T	T	T	F	F
T	T	F	T	F	T	F
T	F	T	F	T	F	F
T	F	F	F	T	T	F
F	T	T	T	T	F	T
F	T	F	T	F	T	T
F	F	T	T	T	F	T
F	F	F	T	T	T	T

The critical row is highlighted. Everywhere that the premises are true the conclusion is true, therefore the argument form is valid.

6 Introduction to Proofs

6.1

Use direct proof to show the following propositions are true:

(a) If the sum of two integers is even then so is their difference.

Solution

Claim: $\forall x, y \in \mathbb{Z}, \text{even}(x+y) \implies \text{even}(x-y)$.

Proof.

$\exists k \in \mathbb{Z} \ni 2k = x + y$	by definition of even: $\text{even}(m) \iff \exists n \in \mathbb{Z} \ni m = 2n$
$2k - 2y = x - y$	algebra: subtract 2y from both sides
$2(k - y) = x - y$	arithmetic: factor 2 out from the left side
$k - y \in \mathbb{Z} \implies \text{even}(x - y)$	by definition of even

$\square$

(b) The sum of any two rational numbers is rational.

Solution

Claim: $\forall x, y \in \mathbb{Q}, x + y \in \mathbb{Q}$.

Proof.

$x \in \mathbb{Q} \implies \exists a, b \in \mathbb{Z} \ni x = \frac{a}{b}$	definition of rational number
$y \in \mathbb{Q} \implies \exists c, d \in \mathbb{Z} \ni y = \frac{c}{d}$	definition of rational number
$x + y = \frac{a}{b} + \frac{c}{d}$	by substitution
$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}$	by rules of arithmetic
$\frac{ad + cb}{bd} = \frac{p}{q},\ p, q \in \mathbb{Z}$	integers closed over multiplication and addition
$\frac{p}{q} \in \mathbb{Q}$	by definition of rational number
$\therefore x + y \in \mathbb{Q}$	by substitution

$\square$

(c) The product of any two consecutive integers is even.

Solution

Claim: $\forall x \in \mathbb{Z}, \text{even}(x(x+1))$.

Proof.

$(\text{even}(x) \wedge \text{odd}(x+1)) \vee (\text{odd}(x) \wedge \text{even}(x+1))$	by Problem 6.2
$\implies \exists m, n \in \mathbb{Z} \ni x(x+1) = 2m(2n+1)$	by definitions of odd and even
$2m(2n+1) = 2(nm+m)$	distributive rule of arithmetic
$nm+m = p \in \mathbb{Z}$	$\mathbb{Z}$ closed under multiplication and addition
$p \in \mathbb{Z} \implies \text{even}(2p)$	definition of even

$\square$

6.2

Use case analysis to prove that: Consecutive integers have opposite parity (parity is whether a number is odd or even)

Solution

Claim: $\forall x \in \mathbb{Z}, (\text{even}(x) \implies \text{odd}(x+1)) \wedge (\text{odd}(x) \implies \text{even}(x+1))$.

Proof.

Case 1: $\text{even}(x) \implies \text{odd}(x+1)$	
$\exists n \in \mathbb{Z} \ni x = 2n$	definition of even
$x+1 = 2n+1$	substitution
$2n+1$ is odd	definition of odd: $2n+1, n \in \mathbb{Z}$
QED for Case 1	
Case 2: $\text{odd}(x) \implies \text{even}(x+1)$	
$\exists k \in \mathbb{Z} \ni x = 2k+1$	definition of odd
$x+1 = 2k+2$	substitution and arithmetic
$2(k+1)$	arithmetic
$2(k+1)$ is even	by definition of even: $2(k+1), k+1 \in \mathbb{Z}$
QED for Case 2	

□

6.3

Use proof by contradiction to prove that there is no greatest integer.

Solution

Claim: $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} \ni x < y$.

Proof. (by contradiction)

Assume not: $\exists x \in \mathbb{Z} \ni \forall y \in \mathbb{Z}, x \geq y$	
Let $y = x+1$	Since y can be any integer we can choose the integer $x+1$
$x \geq y$	by assumption)
$x \geq x+1$	substitution of $x+1$ for y
$\Rightarrow\Leftarrow$	

□