“Chaos: When the present determines the future, but the approximate present does not approximately determine the future.”
- Edward Lorenz

Logistics

- Current Reading:
  - Mitchell, *Complexity: A Guided Tour*, Chapter 2 (By Jan 27th)
  - Flake, The Computational Beauty of Nature, Chapter 10 (By Jan 30th)
  - May, *Simple Mathematical Models with Very Complicated Dynamics* (Jan 27th)

- Homework: Project 1: Dynamical Systems v1.0 is now up.
- Choose a paper and two backups by Friday, Jan 27^{th}. Email your choices to Bianca. You will be pair with whoever chooses or is assigned the same paper. The first presentation will be next Wednesday (Ben is looking for a partner).

Last Time

- We learned a little about difference equations and differential equations.
- We learned Euler’s method for solving differential equations numerically.
- We wrote the pseudocode for a program to execute Euler’s method.

Last Time

- We learned a little about difference equations and differential equations.
- We learned Euler's method for solving differential equations numerically.
- We wrote the pseudocode for a program to execute Euler's method.

Let's try coding it up…

Dynamical Systems and Chaos

Lecture 3

State Space and Phase Space

- Two main types of dynamical systems
  - Differential equations (flows) – Phase Space
  - Iterated maps (difference equations) – State Space
- Framework for ODE: $$\dot{x} \equiv \frac{dx_i}{dt}$$
- State or phase space is the space with coordinates <x_1, … x_n>
- We call this an n-dimensional system, or an n-th order system

$$\dot{x}_1 = f_1(x_1, x_2, \ldots, x_n)$$

$$\vdots$$

$$\dot{x}_n = f_n(x_1, x_2, \ldots, x_n)$$

Chaos

- The logistic *map* was chaotic (discrete time).
- The logistic *flow* was not (continuous time).
- There are several requirements for a system to be possibly chaotic:
    - Contains a non-linear term.
    - In the case of differential equations must have 3 dimensions or more (Consequence of the **Poincaré–Bendixson theorem**).
    - Maps can be chaotic with as few as 1 dimension.
    - The divergence of trajectories is at least exponential.

Linear Systems

- Can be broken down into parts.
- The parts can be solved independently and recombined into a solution.
- A great deal of practical mathematics is based on linearizing systems.
- A change in the initial conditions in a linear system result in a proportional change later on.

Linear vs. Nonlinear

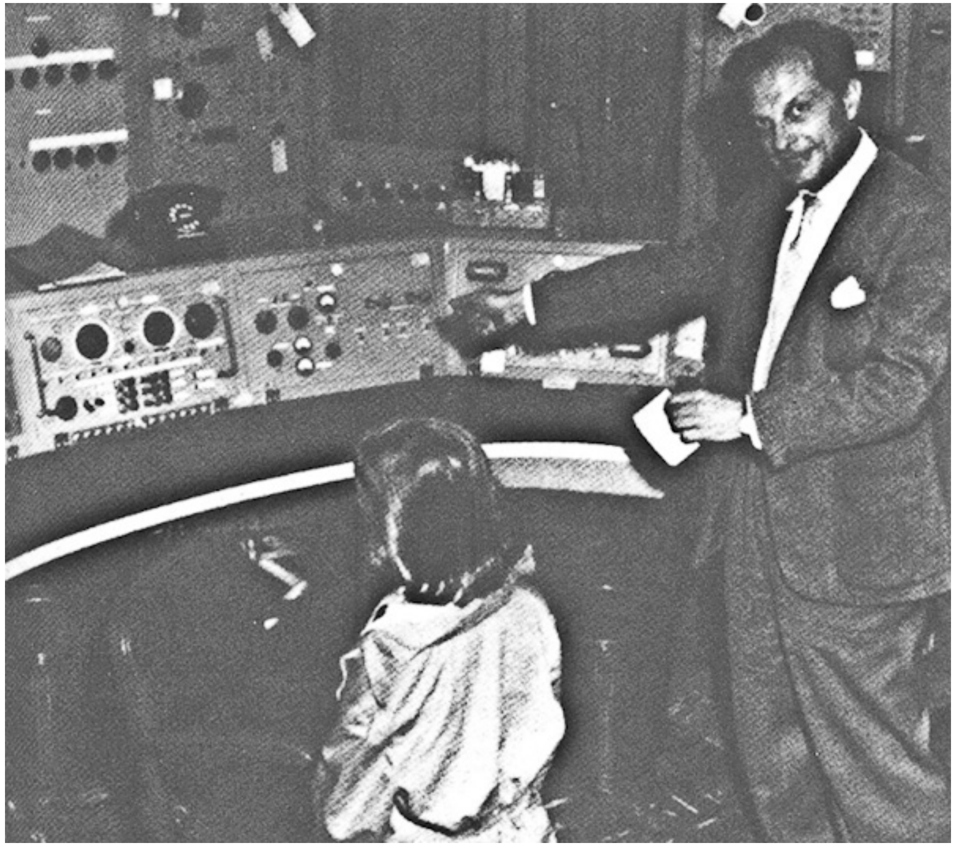
- A system is *linear* if all x_i on the righthand side are of first power only
- Typical nonlinear terms are products, powers, and functions, e.g.
  - X_1x_2
  - $(x_1)^3$
  - $\cos x_2$
- Why are nonlinear systems difficult to solve?
  - Linear systems can be broken into parts and nonlinear systems cannot
- In many cases, we can use geometric reasoning to draw trajectories through state (phase) space without actually solving the system

Non-Linear Systems

- Cannot be broken down into sub-problems like linear systems.
- “Using a term like nonlinear science is like referring to the bulk of zoology as the study of non-elephant animals.”

  \- Stanislaw Ulam.

Most non-linear systems can only practically be studied through computer simulation.



The MANIAC Computer,
Los Alamos National Labs, 1952

Basic Concepts of Dynamics

- State spaces
- Dynamical Systems
- Trajectories (orbits)
- Asymptotic approach to limit sets
- Attractors, basins, and separatrices
- Stability

An aside ;-): Fractals

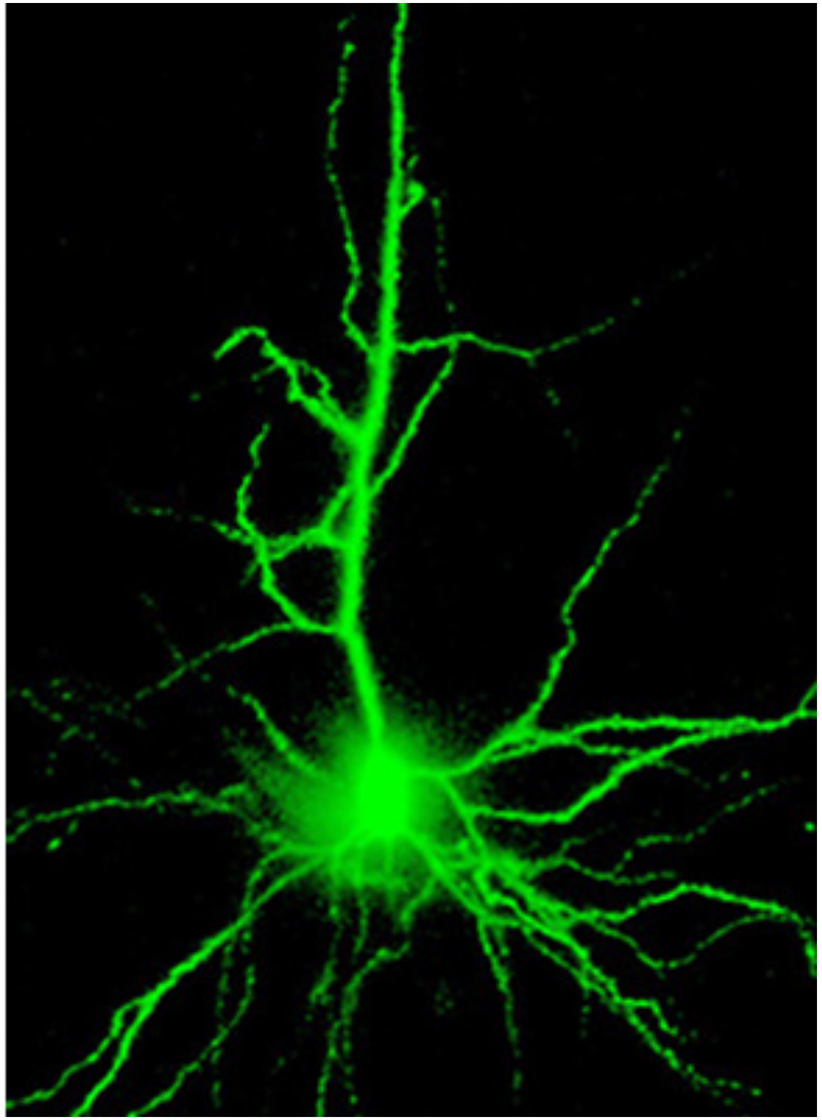
- Nature is not orderly (what does that mean?)
- Fractals are self similar at all scales
- Multifractals are self similar but the similarity can change with scale.
- Simple rules generate fractals but the result seems “unordered”
- Fractals regress infinitely…
- Discussion, what are some possible definitions of
  - Order
  - Complex
  - Chaotic
  - Random

An aside ;-): Fractals

- Nature is not orderly (what does that mean?)

An aside (?): Fractals

- Nature is not orderly
- Keep fractals in the back of your mind.
- Power laws.
- Scale Free.



Logistic Function

- Consider the following differential equation:

$$\dot{x} = \frac{dx}{dt} = rx(1-x)$$

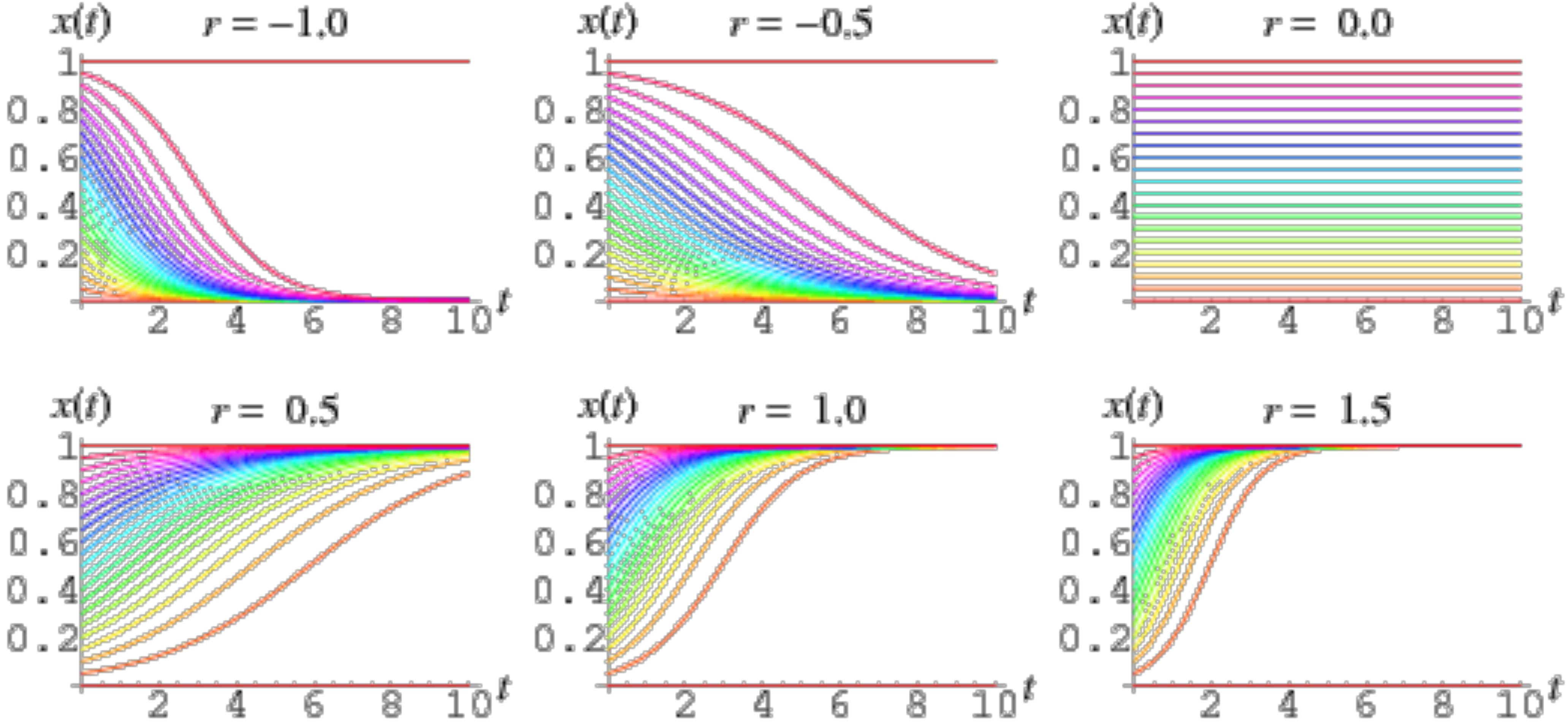
- What happens as we vary r

$$x(t) = \frac{1}{1 + \left(\frac{1}{x_0} - 1\right) e^{-rt}}$$

```
function LogisticFunction( r )

result = [];
h = 0.01;
x0 = 0;
y0 = 0.5;
x_max = 10;

hold on

%for r = 0:0.1:4
F = @(x,y) r*y*(1-y);
result = [];

y = y0;
for x = x0 : h : x_max-h
% The derivative
slope = F(x,y);
% The tangent approximation
y=y+slope*h;

result = [result; y];
end
plot(x0 : h : x_max-h, result);
ylim([0 1.2]);
%end
hold off
end
```

```matlab
function LogisticFunctionODE45(r)
%	right hand side of logistic equation for a matlab numerical
%	solution.
%	r is the intrinsic growth rate

	F=@(x,y) r*y*(1-y);

	tspan=[0 20];
	y0=0.5;
	[t,y] = ode45(F, tspan, y0);
	plot(t,y), xlabel('Time'), ylabel('Population')
end
```

Logistic Function



Wolfram, 2017

Single stable fixed point.

Logistic Map

- Consider the following iterative equation:

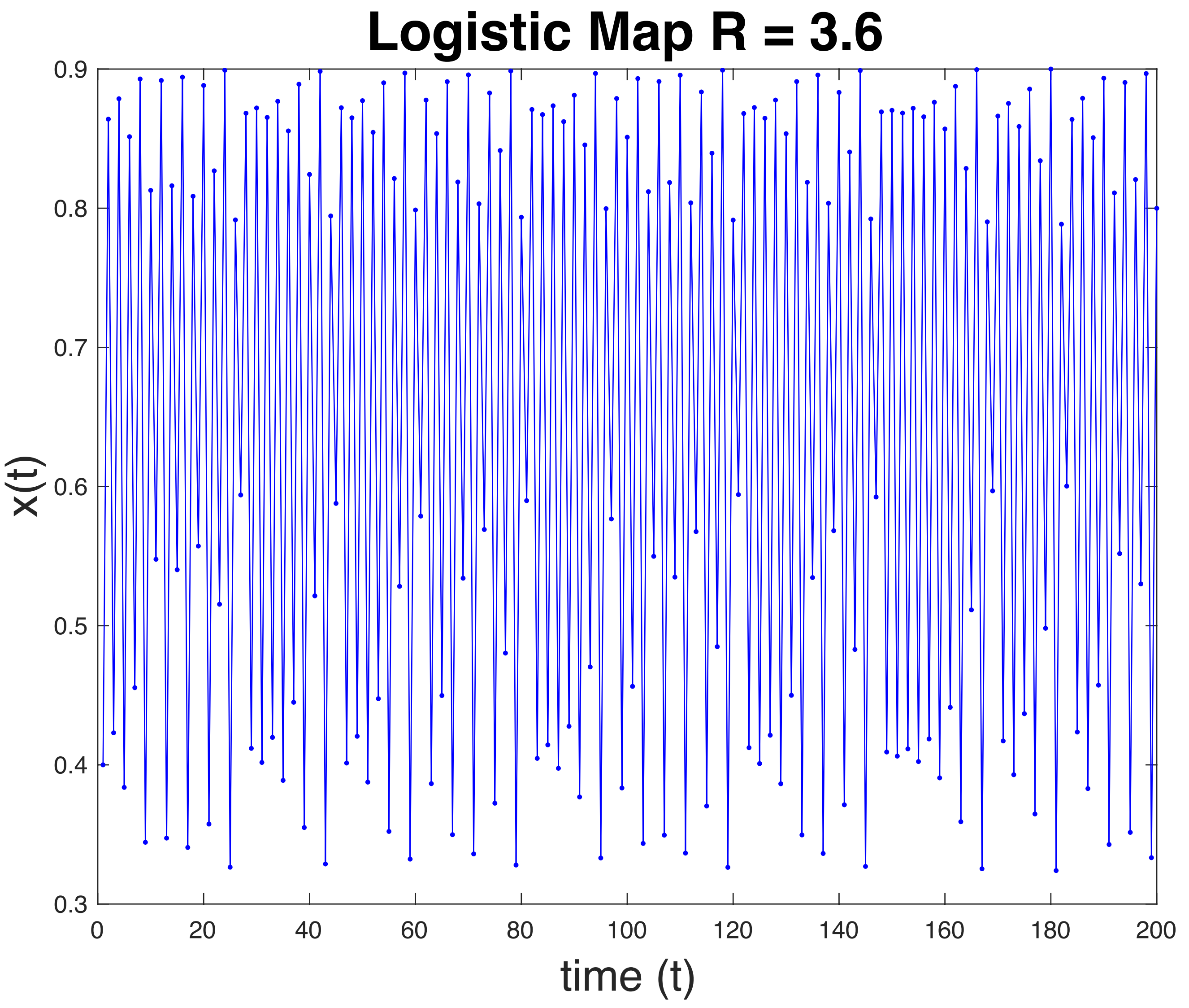
$$x_{t+1} = rx_t(1 - x_t) \quad x_t, r \in [0,1]$$

  - What happens if we vary *r*?

Logistic Map

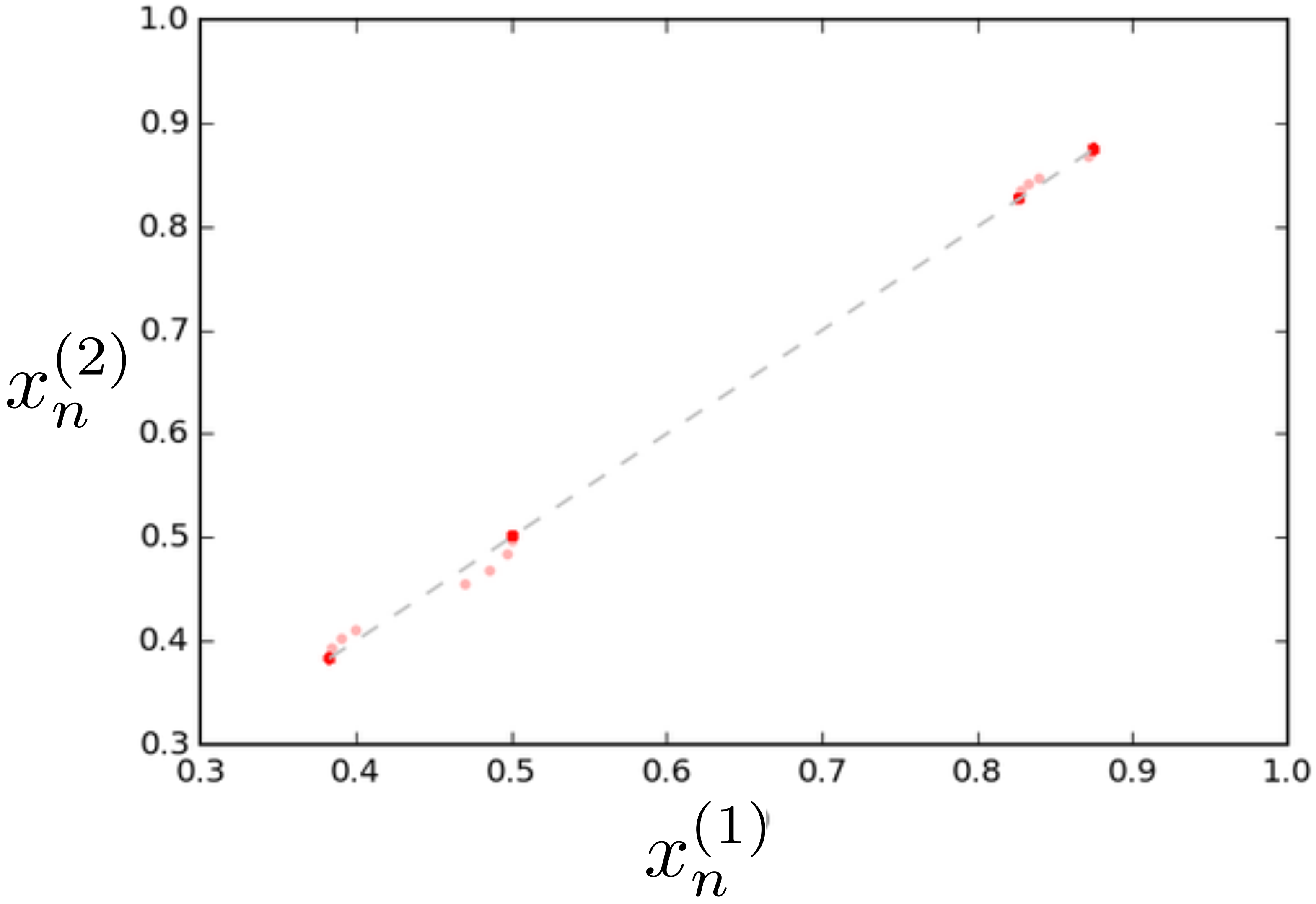
We are interested in the following questions:

- What are the possible asymptotic trajectories given different x_0 for fixed r?
  - Fixed points, limit cycles, chaos
- How do these trajectories change with small perturbations?
  - Stable or unstable



Krivine, Hubert, Annick Lesne, and Jacques Treiner. "Discrete-time and continuous-time modelling: some bridges and gaps." *Mathematical Structures in Computer Science* 17.02 (2007): 261-276.

Logistic Map



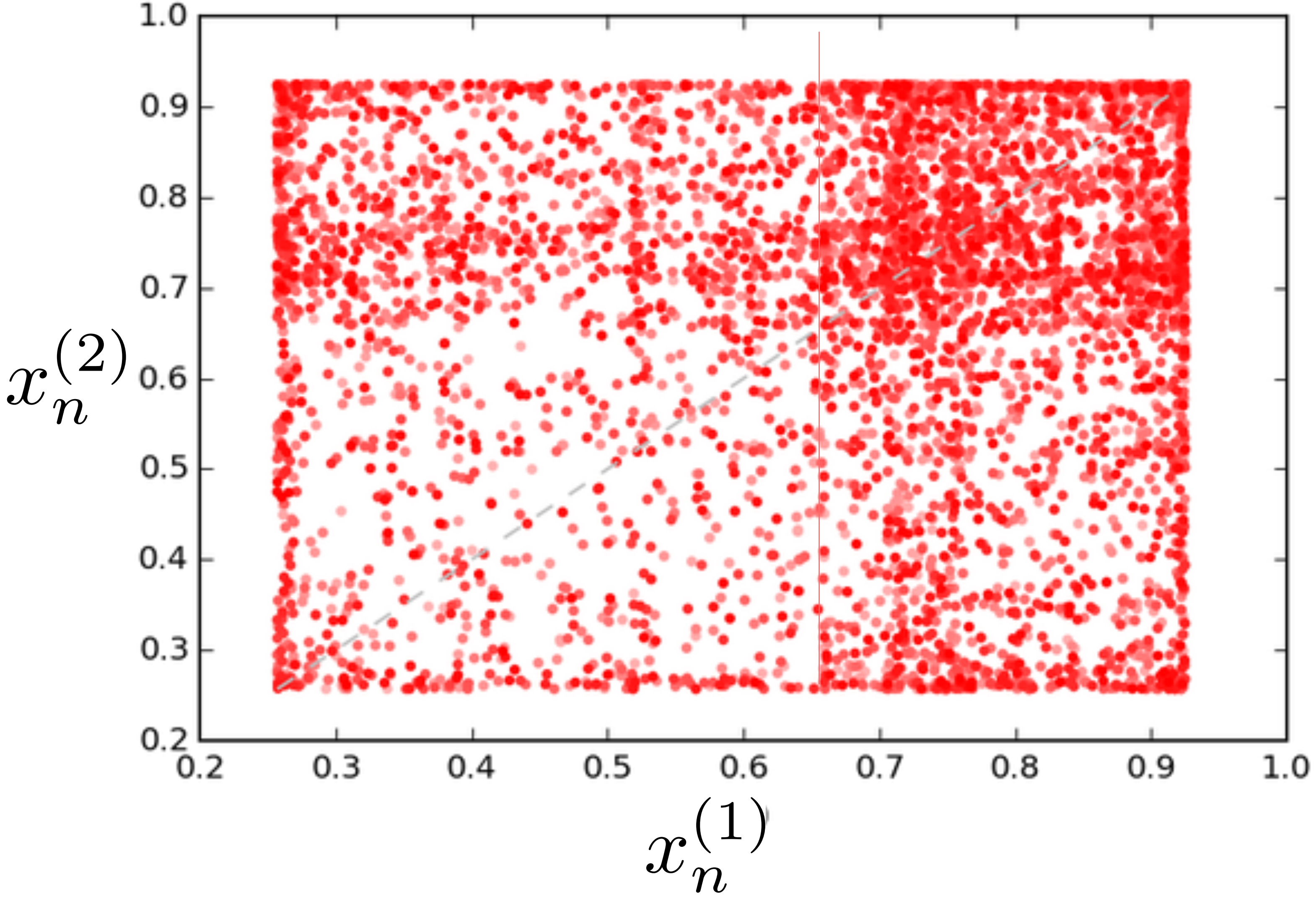
$$x_0^{(1)} = 0.4$$

$$x_0^{(2)} = 0.41$$

$$r = 3.5$$

$$n > \text{ Transient}$$

Logistic Map



$x_0^{(1)} = 0.4$

$x_0^{(2)} = 0.41$

$r = 3.7$

$n >$ Transient

The Logistic Map

a) Time series
b) Y axis is the state at time t, x-axis is the state at time t+1.

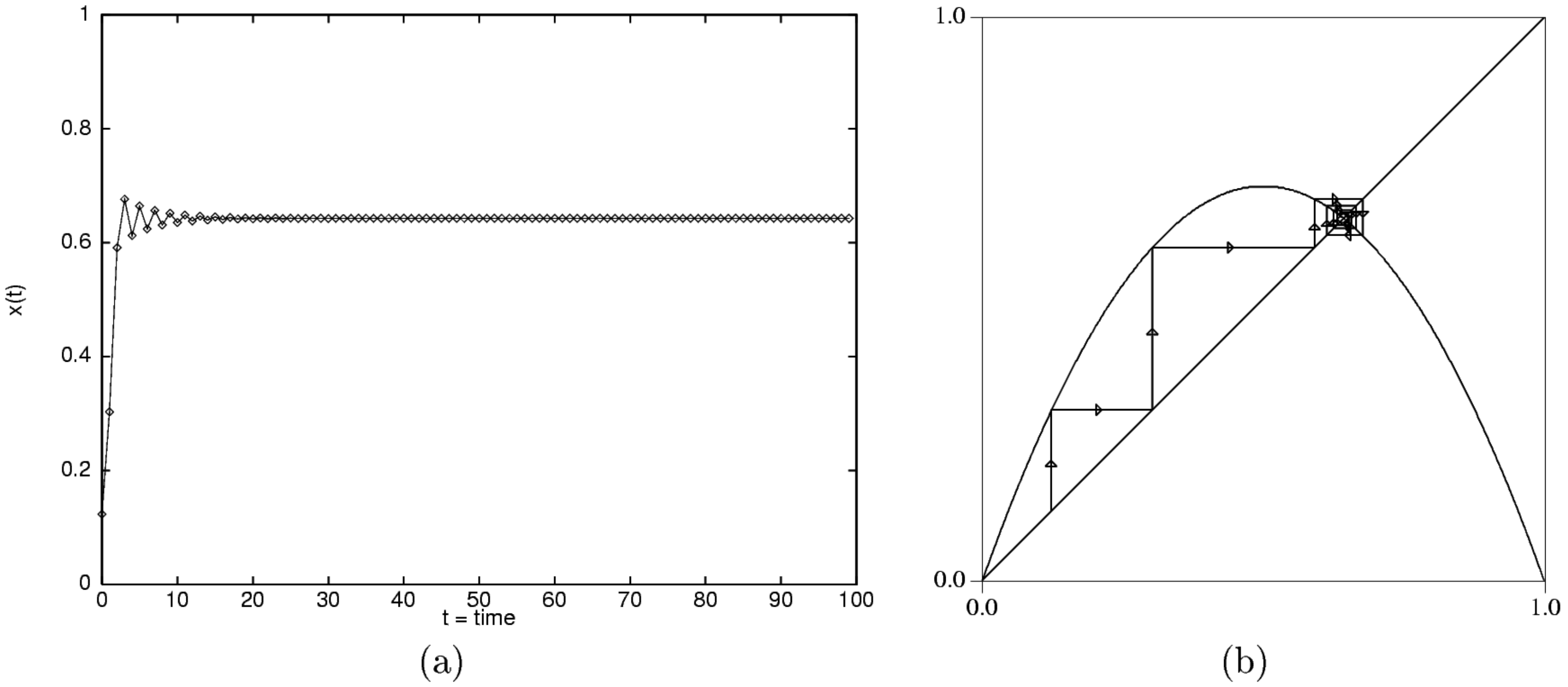


Figure 10.2 Logistic map with $r = \frac{7}{10}$: (a) The time series quickly stabilizes to a fixed point. (b) The state space of the same system shows how subsequent steps of the system get pulled into the fixed point.

Malthus

$$x_{t+1} = 4(7/10)x_t(1 - x_t)$$

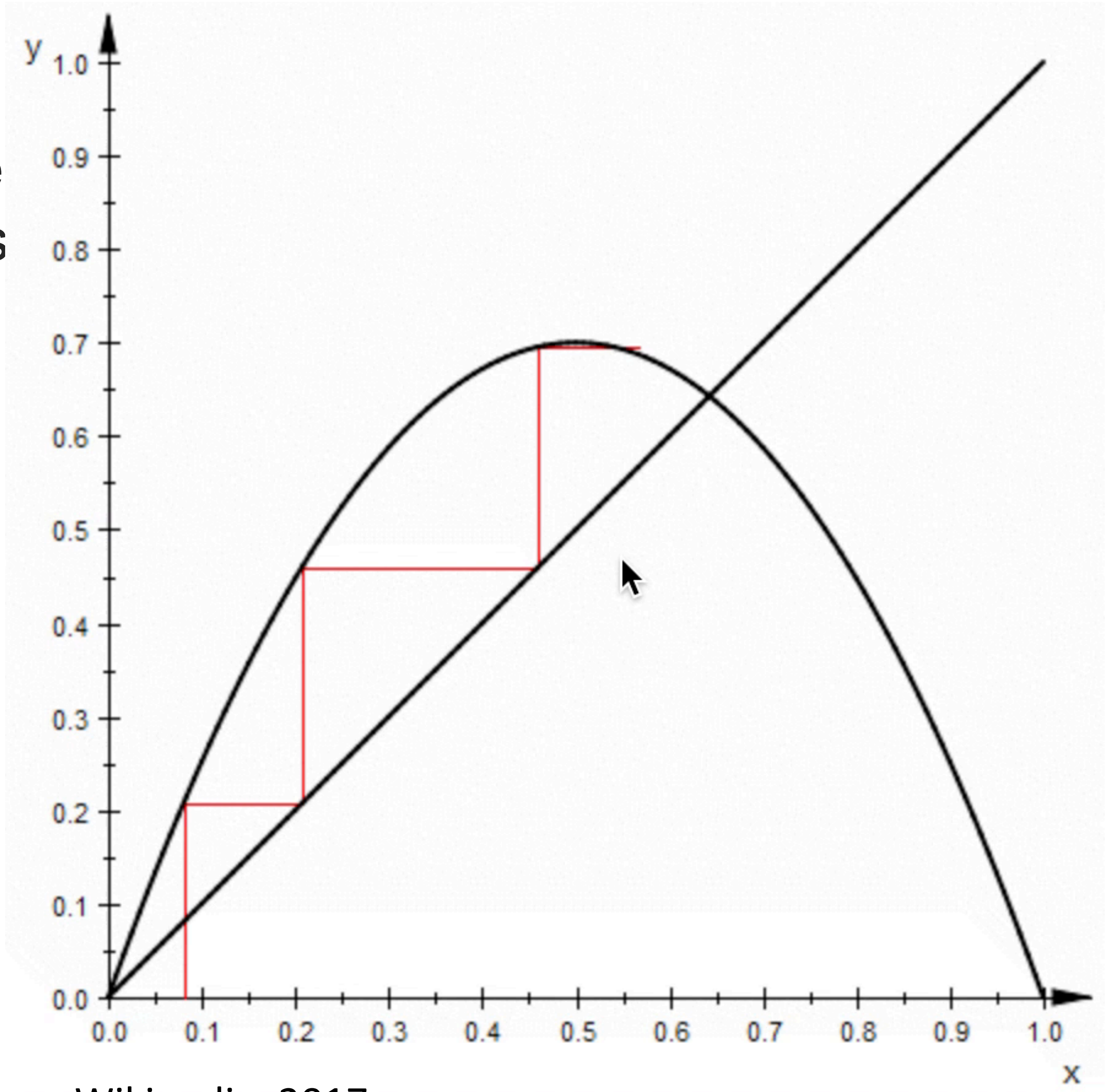
Cobweb Plots

1. Find the point on the function curve with an x-coordinate of x0. This has the coordinates (x0,f(x0)).

2. Plot horizontally across from this point to the diagonal line. This has the coordinates (f(x0),f(x0)).

3. Plot vertically from the point on the diagonal to the function curve. This has the coordinates (f(x0),f(f(x0)).

4. Repeat from step 2 as required.



Wikipedia, 2017

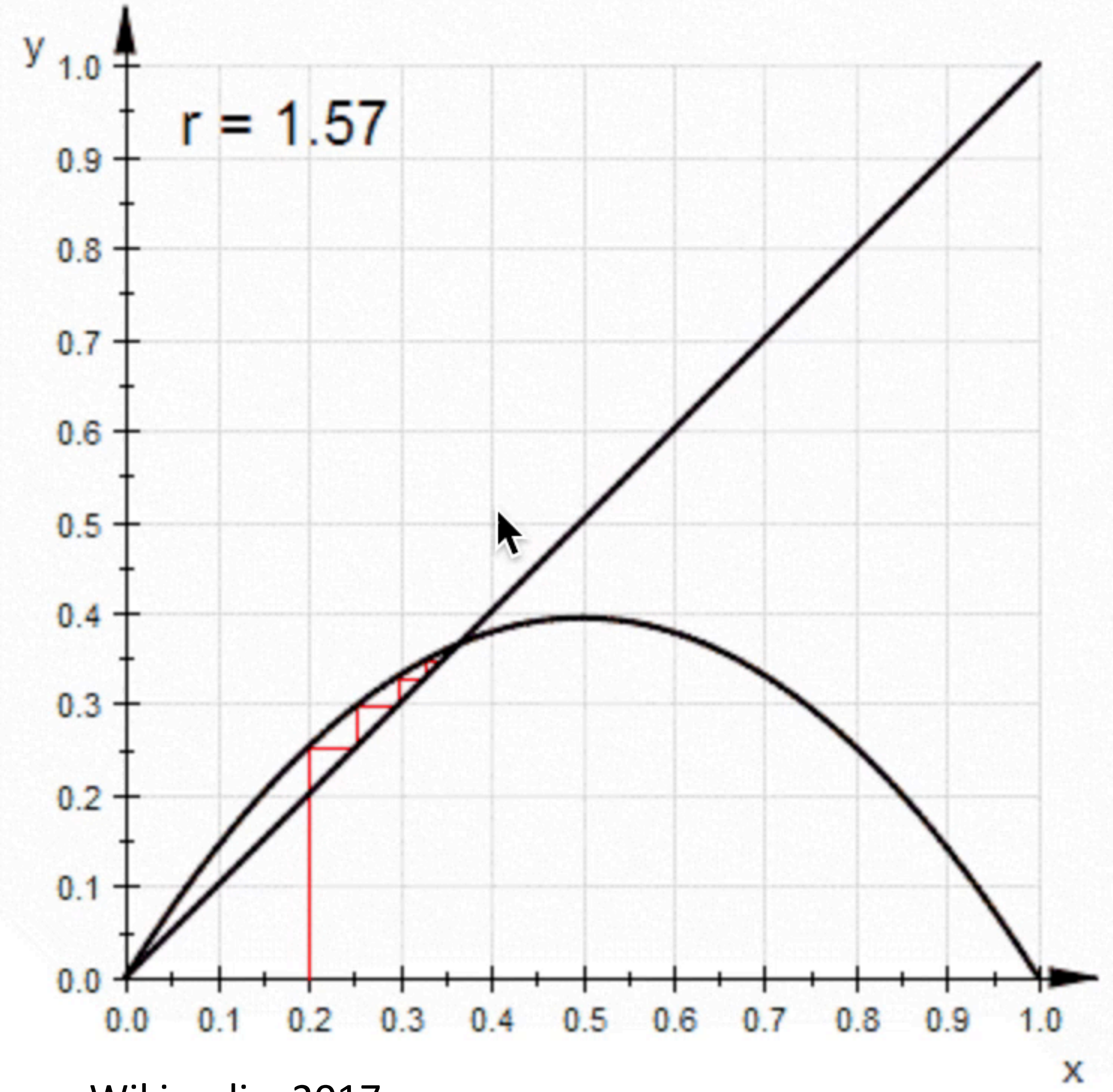
Cobweb Plots

1. Find the point on the function curve with an x-coordinate of x0. This has the coordinates (x0,f(x0)).

2. Plot horizontally across from this point to the diagonal line. This has the coordinates (f(x0),f(x0)).

3. Plot vertically from the point on the diagonal to the function curve. This has the coordinates (f(x0),f(f(x0)).

4. Repeat from step 2 as required.



Wikipedia, 2017

Cobweb Plots

1. Find the point on the function curve with an x-coordinate of x0. This has the coordinates (x0,f(x0)).

2. Plot horizontally across from this point to the diagonal line. This has the coordinates (f(x0),f(x0)).

3. Plot vertically from the point on the diagonal to the function curve. This has the coordinates (f(x0),f(f(x0)).

4. Repeat from step 2 as required.

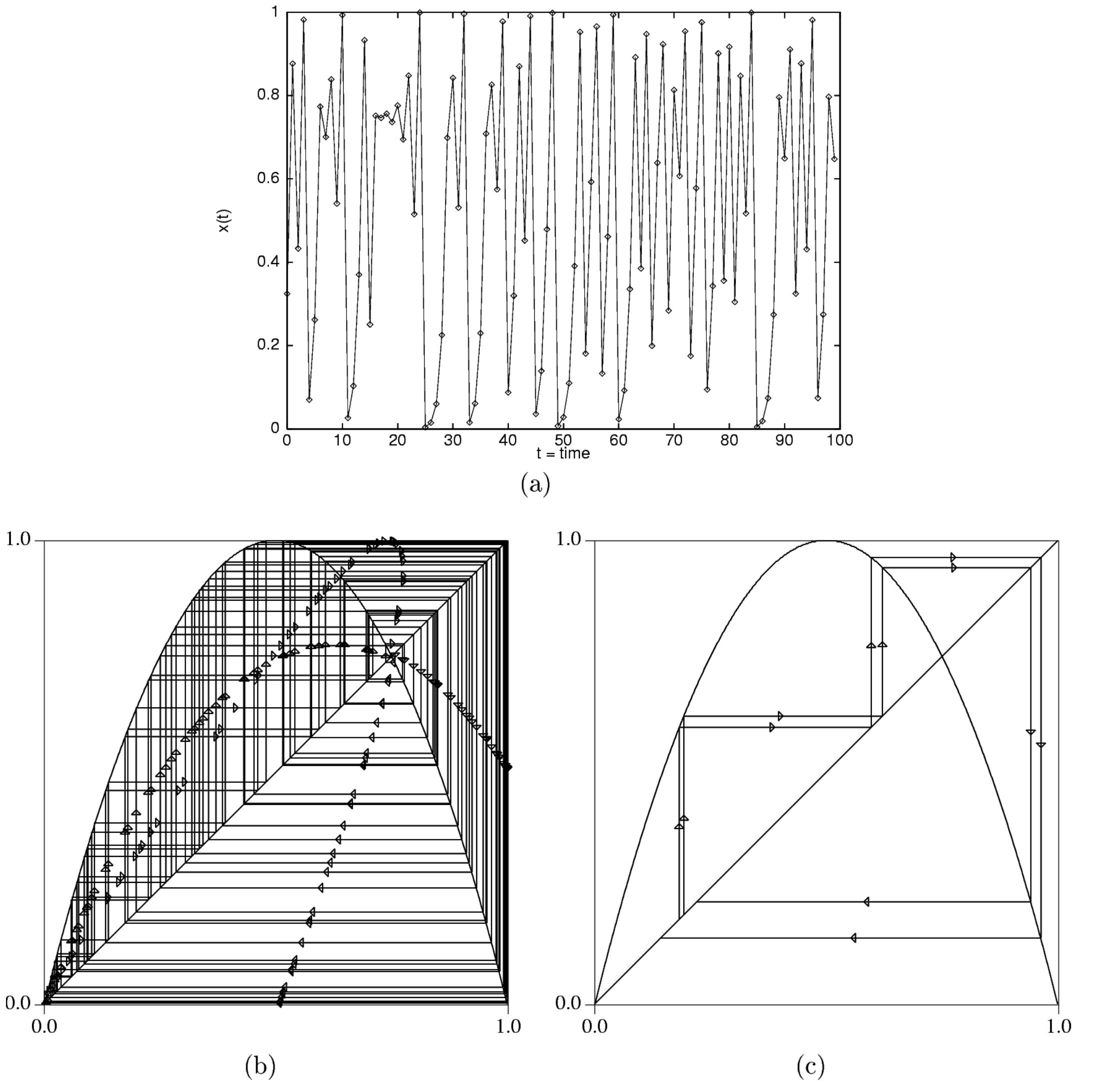


Figure 10.6 Logistic map with $r = 1$: (a) The time series is chaotic and has the appearance of noise. (b) The state space of the same system, which illustrates how the system's trajectory visits every local region. (c) The state space of the same system with only four steps plotted, so as to show how small differences turn into larger differences.

Period Doubling Transition to Chaos

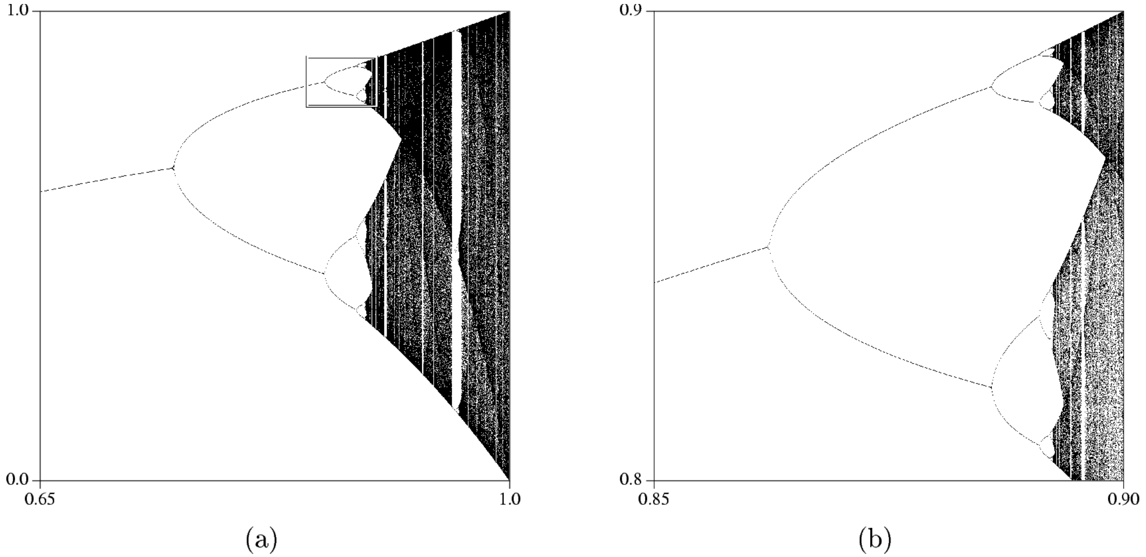


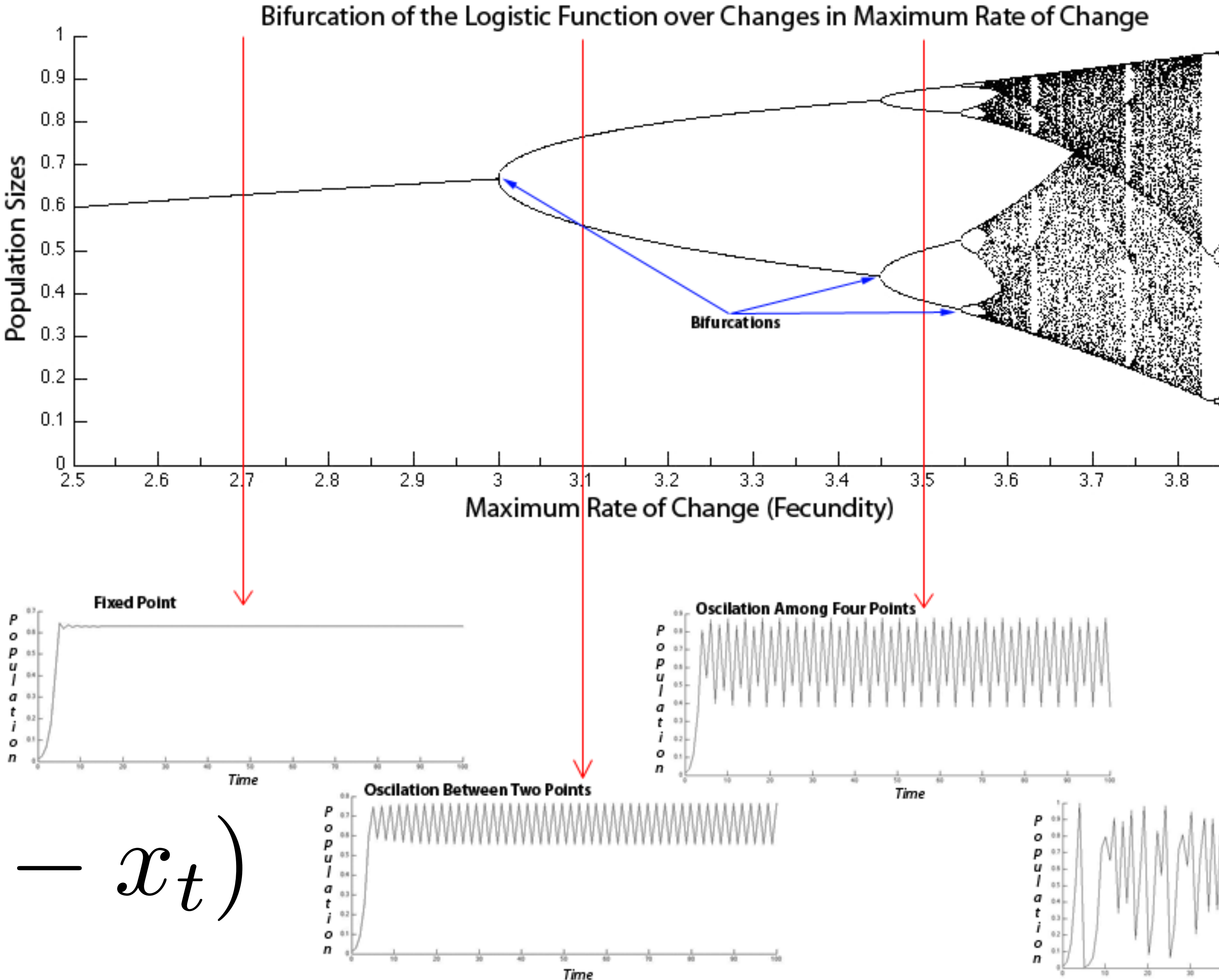
Figure 10.7 Bifurcation diagrams for the logistic map: (a) This image has values of r such that fixed points, limit cycles, and chaos are all visible. (b) This image shows the detail of the boxed section of (a).

X-axis: r
Y-axis: period of limit cycle

- As r increases, between ¼ and ¾:
  - For any given r, system settles into a limit cycle
  - Successive period doublings (bifurcations) as r increases
  - The amount that r increases to get to next period doubling gets smaller and smaller for each new bifurcation. (Feigenbaum's constant)
  - At the critical value, the dynamical system falls into essentially an infinite-period limit cycle

The Logistic Map cont.

- X_t ➔ 0 (stable fixed point), for $r \leq \frac{1}{4}$
- X_t ➔ stable fixed point attractor, for (prev. slide). $\frac{1}{4} < r < \frac{3}{4}$
  note: x_t = 0 is a second fixed point (unstable)
- X_t ➔ periodic with untable points $r > \frac{3}{4}$



$$x_t, r \in [0,1]$$

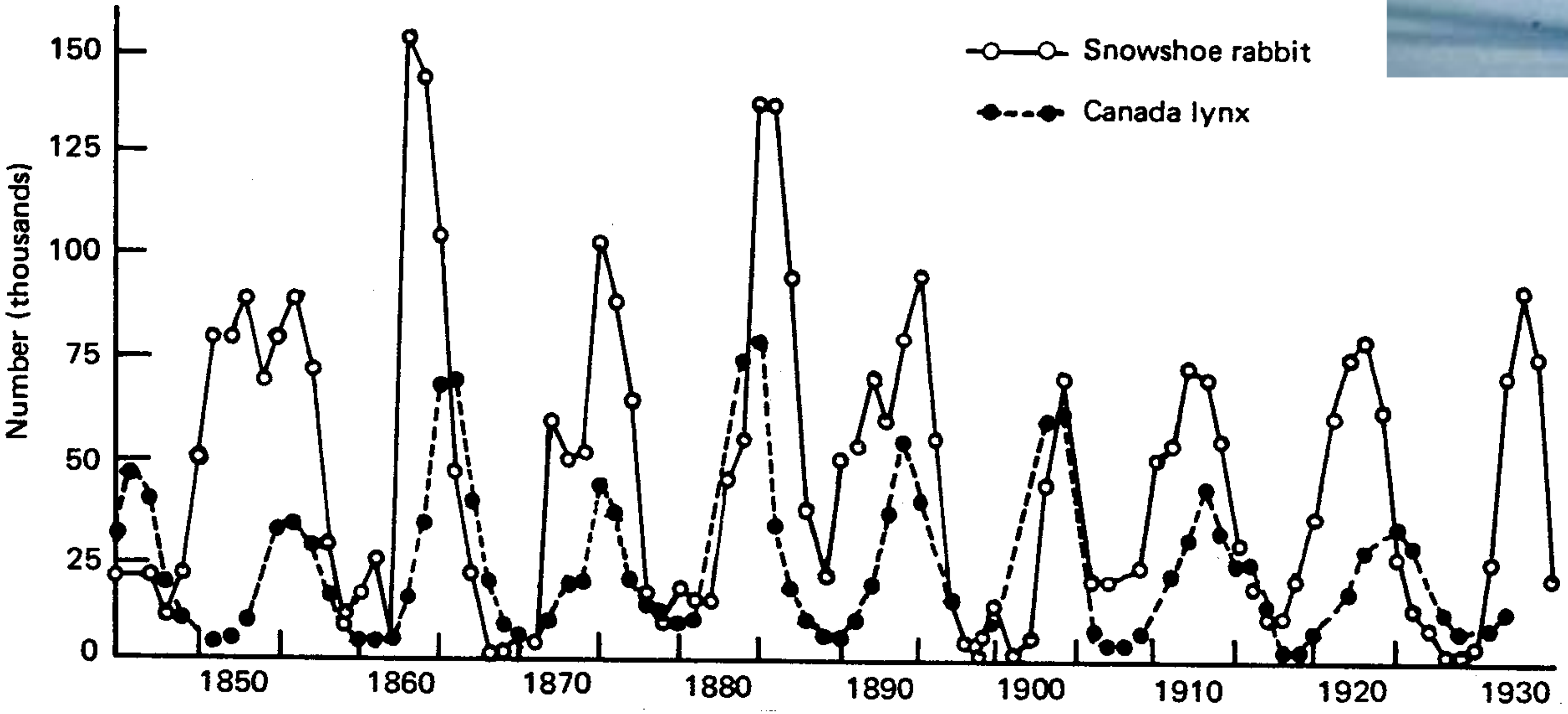
$$x_{t+1} = r x_t (1 - x_t)$$

Coupled Equations: Lotka-Volterra

- The Lotka-Volterra model is the simplest model of predator-prey interactions. It was developed independently by:
  - Alfred Lotka, an American biophysicist (1925), and
  - Vito Volterra, an Italian mathematician (1926).
- Basic idea: Population change of one species depends on:
  - Its current population.
  - Its reproduction rate.
  - Its interactions with other species (predation or prey).
- Model expressed as coupled differential equations:

$$\frac{dx}{dt} = Ax - Bxy \qquad \frac{dy}{dt} = -Cy + Dxy$$

Lynx/Rabbit Historical Data



What are dynamical systems used for?

- Making qualitative predictions about asymptotic behavior of a system
- Example asymptotic behaviors
  - Fixed point
  - Limit cycles and quasi-periodicity
  - Chaotic
- Limit sets: The set of points in the asymptotic limit.

Summary

- A dynamical system:
  - Has a notion of state, which contains all the information upon which the dynamical system acts
  - A simple set of deterministic rules for moving between states (minor exception: stochastic dynamical systems)
  - This is why state diagrams that plot x_t vs. x_{t+1} characterize a dynamical system

Are the digits of pi a dynamical system?

0	1	2	3	4	5	6	7	8	9

3.1415926535897932384626433832795028841971 ...

- Consider the 10 digits as the state space and the deterministic algorithm A that produces digits
- Is this a dynamical system?

Are the digits of pi a dynamical system?

0	1	2	3	4	5	6	7	8	9

3.1415926535897932384626433832795028841971 ...

- Consider the 10 digits as the state space and the deterministic algorithm A that produces digits.
- A digit of pi is not a state of the algorithm (dynamical system) that generates pi.

Are the digits of pi a dynamical system?

0	1	2	3	4	5	6	7	8	9

3.1415926535897932384626433832795028841971 ...

- Consider the 10 digits as the state space and the deterministic algorithm A that produces digits.
- A digit of pi is not a state of the algorithm (dynamical system) that generates pi.
  - The algorithm needs to carry along the history of the computation in order to compute the next digit.
- However, the digits of pi can be produced by a dynamical system

Characteristics of Chaos

- Deterministic
- Unpredictable:
  - Behavior of trajectory is unpredictable in long run
  - Sensitive dependence on initial conditions
- Mixing: The points of an arbitrary small interval eventually become spread over the entire unit interval
  - Ergodic: every state space trajectory will return to the local region of a previous point in the trajectory, for an arbitrarily small region
  - Chaotic orbits densely cover the unit interval
- Embedded: infinite number of unstable periodic orbits within a chaotic attractor
  - No possibility of detecting the periodic orbits by running the time series on a computer (limited precision)
- Bifurcations: repeated branching

Summary

- Bifurcations leading to chaos:
  - The amount by which *r* must be increased to get new period doubling shrinks for each new bifurcation.
  - Until the critical point is reached (transition to chaos)
- Why is chaos important?
  - Seemingly random behavior may have a simple, deterministic explanation
  - Contrast with world view based on probability distributions
- Note: We have not given a formal definition of chaos:
  - We will learn how to find Lyapunov exponents, these are the exponents of the exponential divergence mentioned earlier.
  - Chaos defined by presence of positive Lyapunov exponents and global stability
- Working definition (Strogatz, 1994)
  - “Chaos is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions”

Why is Chaos important?

- Complexity emerges from simple rules
- Vocabulary of complex behavior
- Limits to detailed prediction
  - Shadowing lemma guarantees that for any computed chaotic orbit, there is always a true chaotic orbit that approximates within some ε
  - Implies that statistical properties measured by computer experiments are significant
- Universality
  - E.g., period doubling, Feigenbaum's constant, etc.

Modelling Chaotic Systems

- So what can we do when modelling a chaotic system such as the weather?
- Hint: 50% chance of rain.

Questions About Chaos

- How can we make the notion of chaos precise?
- How can we be sure that what looks like chaos is really chaotic and not just very complicated but perfectly predictable?
  - E.g., When we see a seemingly chaotic time series, how can we be sure that it is not periodic, with an extremely long period?
- How can chaos be measured?
- What is the value of numerical calculations in the presence of chaos?

References

- The Computational Beauty of Nature (Ch. 5-6)
- Chaos and Fractals by by H. Peitgen, H. Jurgens, and D. Saupe. Springer-Verlag (1992).
- Nonlinear Dynamics and Chaos by S. H. Strogatz. Westview (1994).
- J. Gleick *Chaos*. Viking (1987).
- Robert L. Devaney *An Introduction to Chaotic Dynamical Systems.* Addison-Wesley (1989).
- Ralph Abraham and Christopher D. Shaw *Dynamics-The Geometry of Behavior* Vol. 1-3 (1984).
- Stephanie Forrest for sharing her slides from previous classes.
- Tiberiu Tesileanu, https://www.quora.com/Chaos-Theory-What-is-the-difference-between-chaotic-behavior-and-random-behavior

Discussion Questions

- What, if anything, does all this have to do with fractals?
- What, if anything, does this have to do with computation?
- Are the digits of pi a dynamical system?
- Problems of computing chaotic orbits on a computer?

Flows and Maps

- Question about the relationship between Euler's discrete approximation of a flow a discrete time map.

- It is possible, for certain values of step size h to reproduce the logistic map using the logistic function ($h=0.67$).