Cellular Automata as Models of Complexity

Stephen Wolfram, 1984

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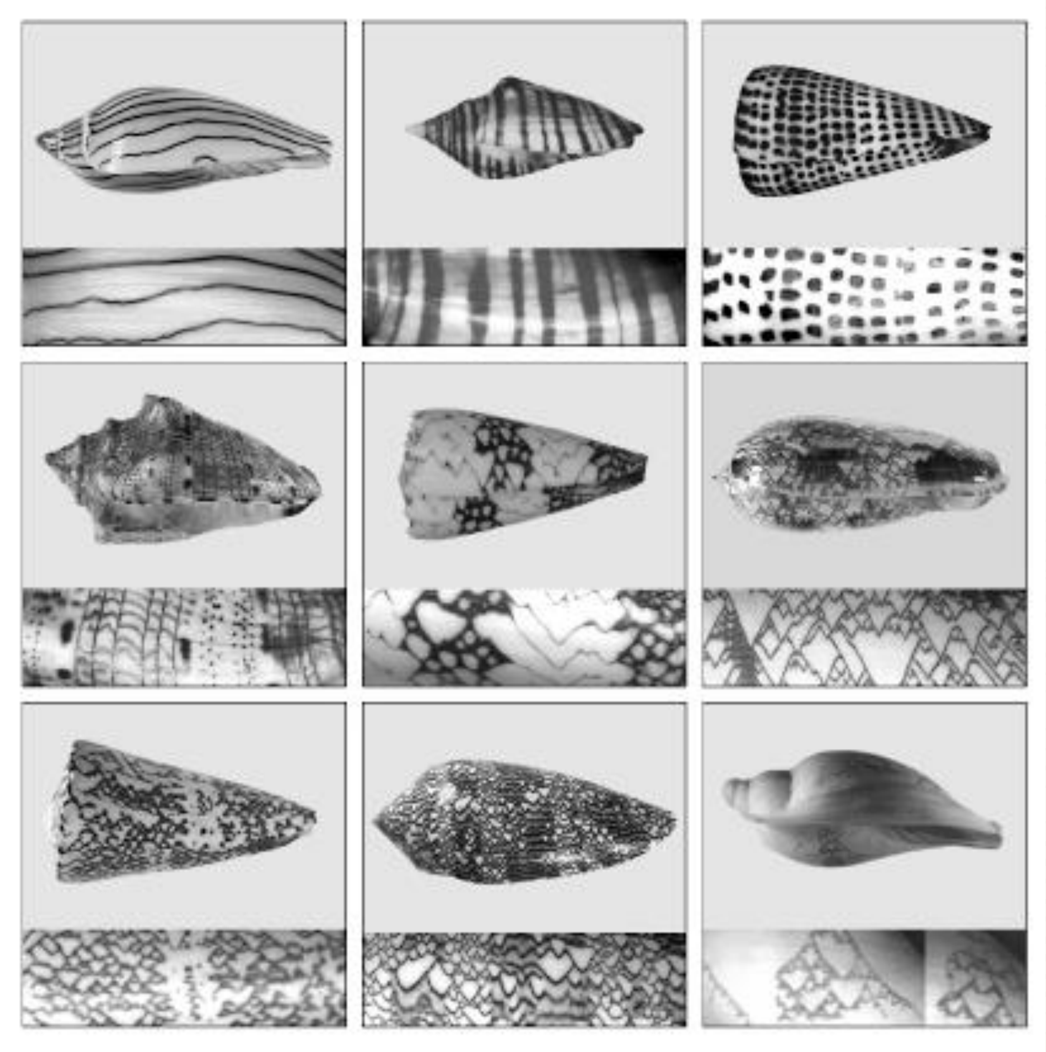
Introduction

- Cellular Automata
- Applications
- Types of Cellular Automata
- Irreversibility
- Limiting Entropy
- Information Propagation
- Formal Language Theory
- Complexity
- Computation Theory
- Thermodynamics



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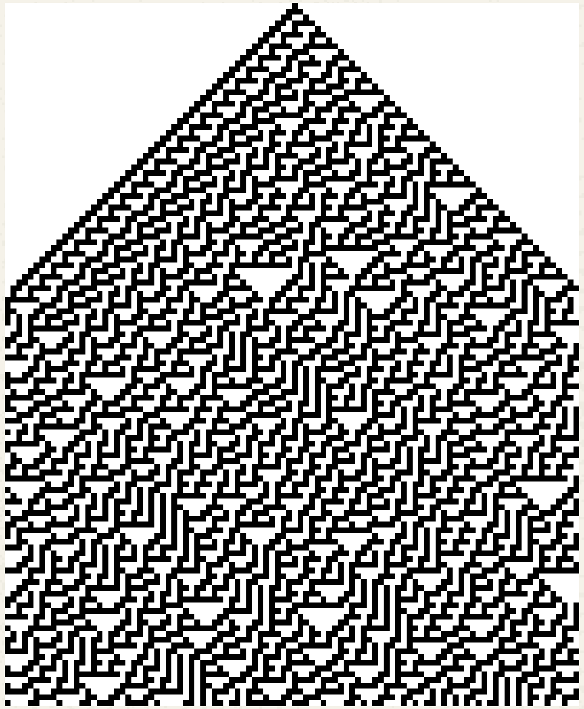
Cellular Automata

A mathematical system constructed from many identical components.

A grid of binary values (or values 0,...,k-1) that update according to a deterministic rule depending on a neighborhood

Complex behavior can arise from simple rules

$$a_i^{(t+1)} = \phi[a_{i-r}^{(t)}, a_{i-r+1}^{(t)}, \ldots, a_{i+r}^{(t)}]$$



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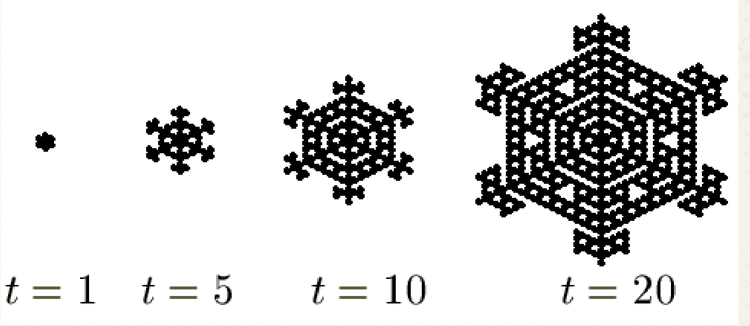
Applications

Cellular automata may serve as suitable models for a wide range of natural systems such as:

- The growth of dendritic crystals (snowflakes)
- Pigmentation patterns on mollusk shells

This paper concentrates on general mathematical features of behavior – not specific applications



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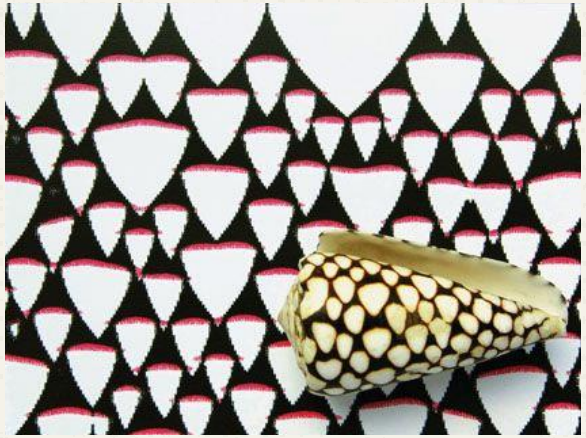

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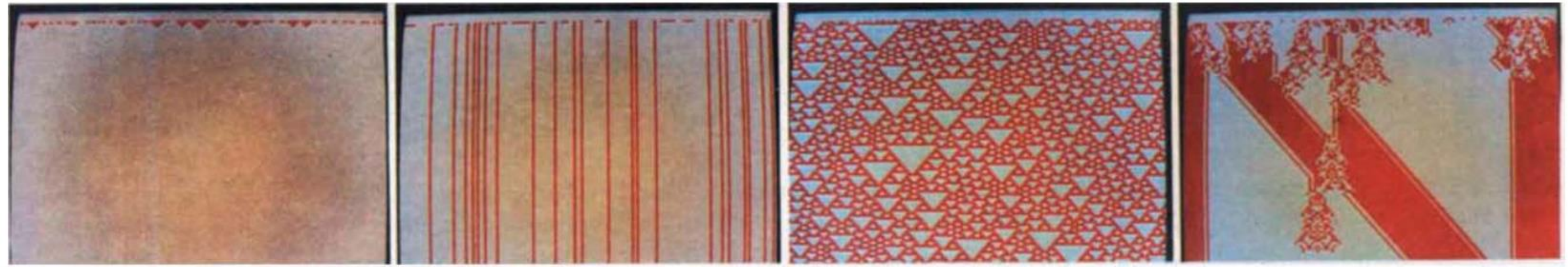
Cellular Automata

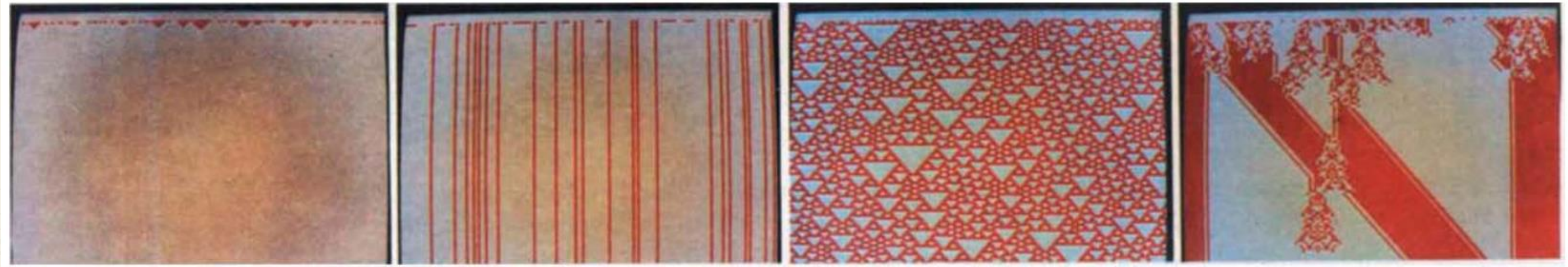
Four classes of automata:

1. Disappears with time – Fixed homogenous state
2. Evolves to a fixed finite size – Limiting cycles
3. Grows indefinitely at a fixed speed – chaotic, aperiodic pattern
4. Grows and contracts irregularly – Complex localized structures



Irreversibility

Different configurations may evolve to the same configuration, thus initial configurations cannot be recalculated



Limiting Entropy (Dimension)

A generalized Measure of the density of configurations generated

Limiting Entropy = $$d^{(x)} = \lim_{X \to \infty} \frac{1}{X} \log_k N(X)$$

N(X) gives the number of distinct sequences of X site values that appear

d(x) = 0 for class 1 and 2 due to them reaching a limit set

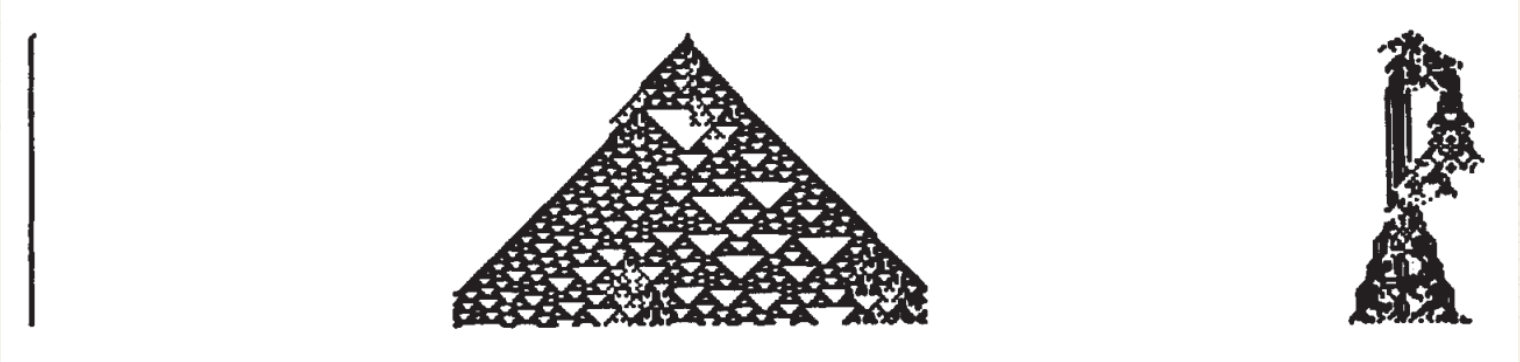
For class 3 the limiting entropy decreases in time, suggesting a fractal subset of all possible configurations occurs

Information Propagation

Characterized by the stability of predictability of their behavior under small changes in initial conditions

Information propagation rates:

- Finite distance in class 1 and 2
- Infinite distance at fixed speed in class 3
- Irregular propagation over an infinite range in class 4



Information Propagation

- The degree of sensitivity to initial conditions shows us degrees of predictability for the cellular automation:

- Class 1 – Entirely predictable, independent of initial state
- Class 2– Local behavior predictable from local initial state
- Class 3 – Behavior depends on an ever-increasing initial region
- Class 4 – behavior effectively unpredictable

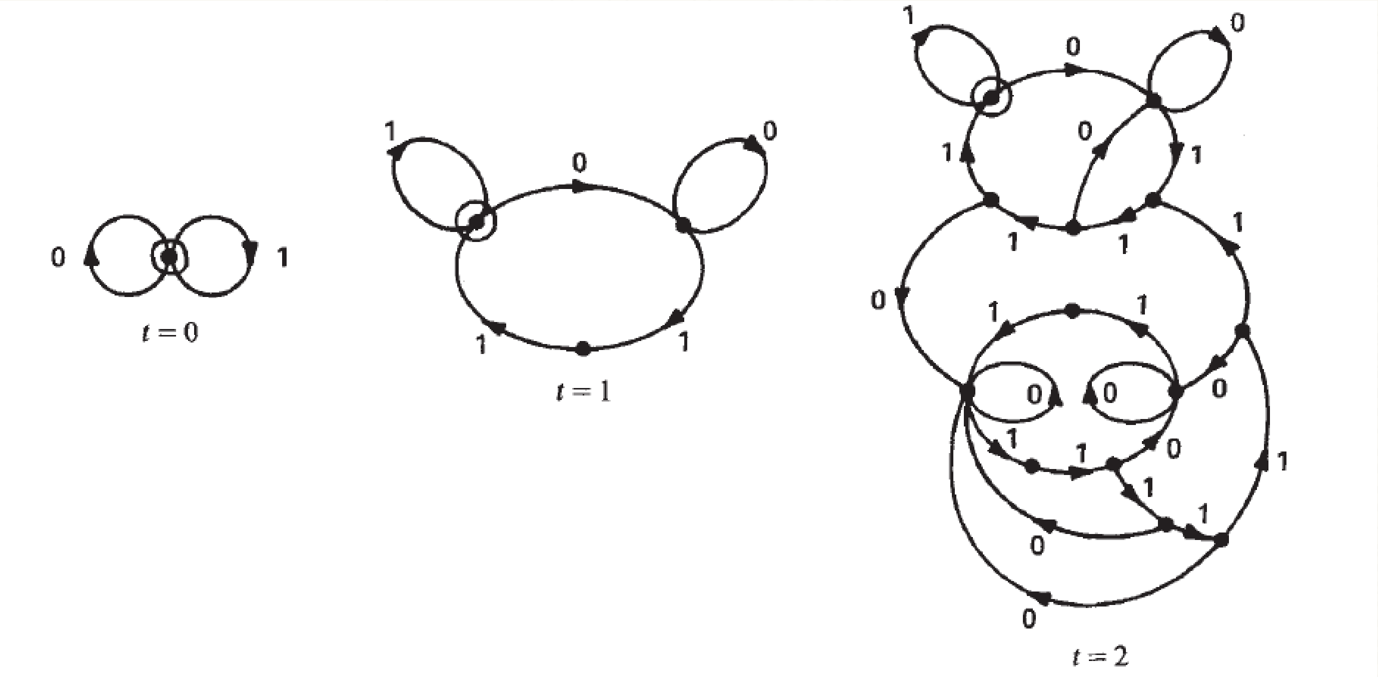
Formal Language Theory

- What is formal language?
  - Set of symbols/strings over some finite alphabet + set of rules (grammar).
  - Recognized by idealized computers with a CPU containing a finite number of internal states + memory
- Types of formal languages:
  - Regular languages (computation model -> Finite State Automaton): No memory required
  - Context-free languages (computation model -> Non-deterministic Pushdown Automaton): Need LIFO stack
  - Context-sensitive languages: memory size = input word size
  - Unrestricted languages (Turning Machine): Large memory required

Why Formal Language Theory?

- Quantities like entropy and dimension (Information theory) give rough characterization of cellular automaton behavior.
- Computation theory gives complete description of self-organization in cellular automaton
- Sets of cellular automaton configurations = Formal Languages
  - Consists of symbols (site values) + set of rules (grammar)
  - Set of all possible initial configurations = Trivial formal language
  - Set of configurations after any finite number of time steps = Regular language

Formal Language Theory



- Set of configurations generated in the first few time steps of evolution according to a class 3 cellular automaton rule.

Observations from graph

- Possible configurations = possible paths, beginning at the encircled node.
- All possible configurations are allowed at t=0. As time grows, less configurations are generated.
- Set of all possible configurations at each time step forms regular language.
- Number of nodes in the smallest graph = Regular language complexity of the set of configurations.
  - Larger the number of nodes = more complicated sets. E.g. t=3, complexity=107 and t=4, complexity = 2867

Complexity

- Regular language complexity for the sets generated by cellular automaton evolution is non-decreasing with time.
  - Complexity represents fundamental property of self-organizing systems because higher complexity means increasing self-organization.
- Class 1 and 2 cellular automata limiting sets = Regular languages
- Class 3 and 4 cellular automata limiting sets != Regular languages.
  - Class 3 cellular automata = context-sensitive languages.
  - Class 4 cellular automata = unrestricted languages(TM).

Computation theory

- Cellular automaton evolution can be viewed as computation.
- Dynamic systems theory can be used to define class 1 ,2, and 3 cellular automata; but not for class 4.
- Computational theory needed to define class 4 cellular automata.

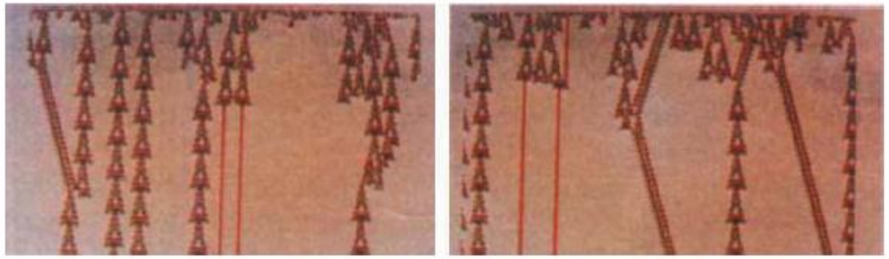


Fig. 5 Examples of the evolution of a typical class 4 cellular automaton from disordered initial states. This and other class 4 cellular automata are conjectured to be capable of arbitrary information processing, or universal computation. The rule shown has $k = 3$, $r = 1$, and takes the value of a site to be 1 if the sum of the values of the sites in its three-site neighbourhood is 2 or 6, to be 2 if the sum is 3, and to zero otherwise (totalistic code 792).

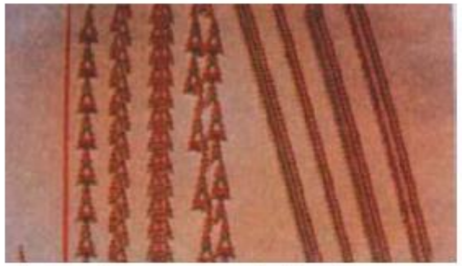


Fig. 6 Persistent structures generated in the evolution of the class 4 cellular automaton of Fig. 5. The first four structures shown have periods 1, 20, 16 and 12 respectively; the last four structures (and their reflections) propagate: the first has period 32, the next three period 3, and the last period 6. These structures are some of the elements required to support universal computation.

- Class 4 cellular automata rules can be used to implement arbitrary information processing operations.
    - Capable of universal computation
        - Evolution can implement any finite algorithm for a given initial conditions

Computation theory

- A short-cut needed for computing the outcome of cellular automaton evolution:
  - Class 1 and 2 need simple computation for future prediction
  - Class 3 and 4 may not allow short-cuts
    - Explicit observation or simulation needed for future prediction.
    - For class 4, simulation is needed for almost all initial configurations. For class 3, some initial configurations might not need simulation.
    - Infinite time limiting behavior will be undecidable (halting problem).
    - Large time limit of entropy can't be computed.

Computation theory

- For class 3 and 4:
  - the occurrence of a particular length n site value sequence in infinite time is undecidable.
  - Decidable in finite time considering initial sequence length n0= n+ 2rt
    - Large n or t will make computation difficult.
- Finding possible sequences generated by class 3 and 4 is an NP-complete problem.
  - Both NP an NP-hard
  - Can't be solved in any time polynomial in n or t.
- Undecidability and intractability are common problems in almost all non-linear systems.
  - No simple formulae for the behavior of such systems (e.g. natural systems)
  - Only simulation and observation are effective for future prediction.

Thermodynamics

- Entropy in conventional thermodynamics (reversible evolution):
  - Fine-grained:
    - Remains constant with time
  - Coarse-grained:
    - Always non-decreasing with time
    - Very little impact on the measures of self-organization structures in long time

$$d_\mu^{(x)} = -\lim_{X\to\infty} \frac{1}{X} \sum_{j=1}^{k^x} p_j \log_k p_j$$

- Set of configurations with low coarse-grained entropy
  - Value of every 5th site highly constrained.
  - Arbitrary value for the intervening sites allowed

Thermodynamics

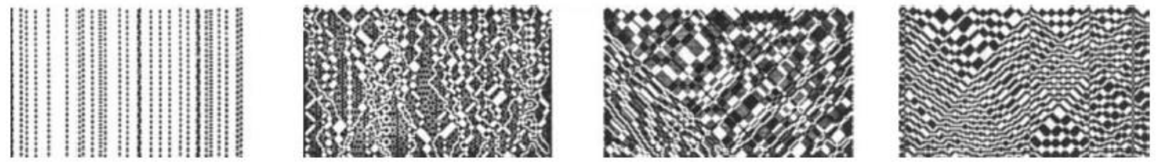


Fig. 7 Evolution of some cellular automata with reversible rules. Each configuration is a unique function of the two previous configurations. (Rule numbers 4, 22, 90 and 126 are shown.) As initial conditions, each site in two successive configurations is chosen to have value 1 with probability 0.1.

- Cellular Automata Rules:
  - Reversible:
    - Conventional Thermodynamics can be applied
    - Every configuration has a unique predecessor in evolution
    - Spatial entropy and dimension remain constant with time
  - Irreversible:
    - Spatial entropy and dimension decrease with time
    - Coarse-grained entropy typically increases for a short time, but then decreases to follow the fine-grained entropy

References

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Questions?

Thank you

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