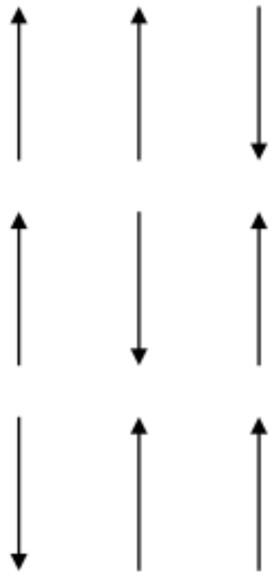
# Self-organised criticality, catastrophe theory, and computation at the edge of Chaos

Lecture 12

# Ising Spin Model



Model of ferromagnet - collection of magnetic moments associated with the spins of atoms.

Treat the spins as located at fixed points on a 2D lattice. Ignore quantum mechanical effects (active area of research).

Consider spin projections along either +z or -z direction.

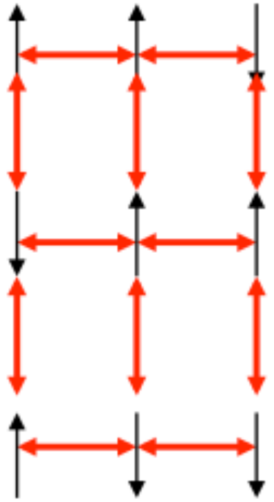
Consider only nearest neighbor interactions.

See:
Computational Physics,
N. Giordano, H. Nakanishi,
2nd Ed., McGraw Hill

$$E = -J\sum_{\langle ij\rangle} s_i s_j \qquad s_{i,j} = \pm 1$$

where $\langle ij\rangle$ means all nearest neighbor spins

# Ising Spin Model



$$E = -J\sum_{\langle ij \rangle} s_i s_j$$

$s=-1 \quad s=1$

$E = J$

$s=1 \quad s=1$

$E = -J$

 is a link. There are

$$(N_{rows}-1)N_{columns} + (N_{columns}-1)N_{rows} = 2N_{columns}N_{rows} - N_{rows} - N_{columns}$$

links. Since simulations of large arrays is limited by computer time, we often employ periodic boundary conditions, where we also have a link between spins in rows $1,N_{rows}$, and spins in columns $1,N_{columns}$. In this case, there are $N_{columns}$*$N_{rows}$ links.

# Statistical Mechanics

In the spirit of Statistical Mechanics, we consider the energy associated with a spin state (which corresponds to knowing the spin directions of all atoms on the lattice). For a system in equilibrium with a heat bath, the probability associated with being in a given state depends only on the energy:

$$P_\alpha \propto e^{-E_\alpha / kT}$$

For a square lattice with $N$ sites per direction, there are $N^2$ sites and a total of $2^{N^2}$ possible states. Even a small lattice with $N=10$ is impossible to study with 'brute force'. On the other hand, such a small lattice is very far from typical real systems, where we deal with $O(10^{23})$ spins. Boundary conditions in the simulation become important.

# Statistical Mechanics

Typical quantities of interest are:

$$< E >= \sum_{\alpha} P_{\alpha} E_{\alpha} \qquad < M >= \sum_{\alpha} P_{\alpha} M_{\alpha}$$

where $M_{\alpha} = \sum_{i} s_i$

Note that there are often many 'microstates' - here, arrangement of spins - with the same energy. The normalization of the probability is

$$P_{\alpha} = \frac{e^{-E_{\alpha}/kT}}{\sum_{\alpha} e^{-E_{\alpha}/kT}}$$

Competition between highest probability state (min energy) and large number of lower probability states (max entropy). Balance controlled by the temperature, T.

# Monte Carlo Algorithm

- Update scheme heuristic.
- We have various choices about how to update a CA.
- Usually we update all the states of the CA simultaneously (synchronous)
- We can also update the states asynchronously. Time appears to pass at different rates for different cells.
- If we update the cells uniform randomly time passes at the same expected rate everywhere.
- This is often much faster than updating everything all the time and converges to most likely outcome.

# Metropolis Algorithm

Nicholas Metropolis, Manhattan Project Computation Lead
Los Alamos National Labs
Supervised John von Neumann and Stanislaw Ulam.
Worked on the MANIAC I computer.

“Johnny von Neumann who was very, very quick—I mean, you have no idea how quickly he would infer things and extrapolate them. Well, he was fantastic.”

...the method we employ is actually a modified Monte Carlo scheme, where, instead of choosing configurations randomly, then weighting them with $\exp(-E/kT)$, we choose configurations with a probability $\exp(-E/kT)$ and weight them evenly.

Nicholas Metropolis, Arianna W. Rosenbluth, Marshall N. Rosenbluth, Augusta H. Teller, and Edward Teller. Equation of state calculations by fast computing machines. *The Journal of Chemical Physics*, 21(6):1087–1092, 1953.

# Metropolis Algorithm

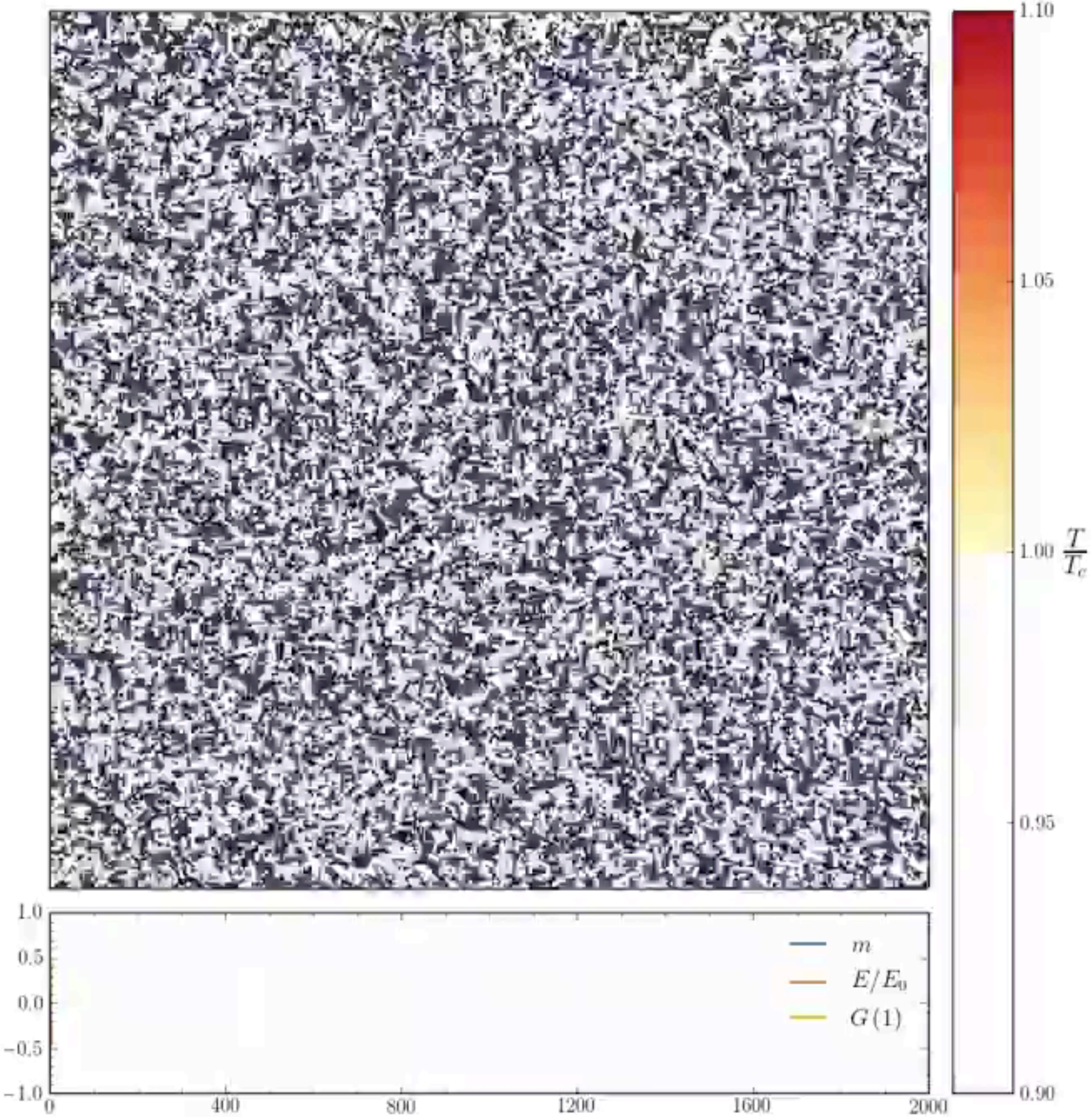
Update our model randomly but only accept the new state with probability
Proportional to the probability of the state really happening.

For example: a particle might try to jump *k* positions in the lattice from its current position but takes energy to do so. Let's say the energy required
Is linear with the distance travelled up to a maximum of 10 steps.

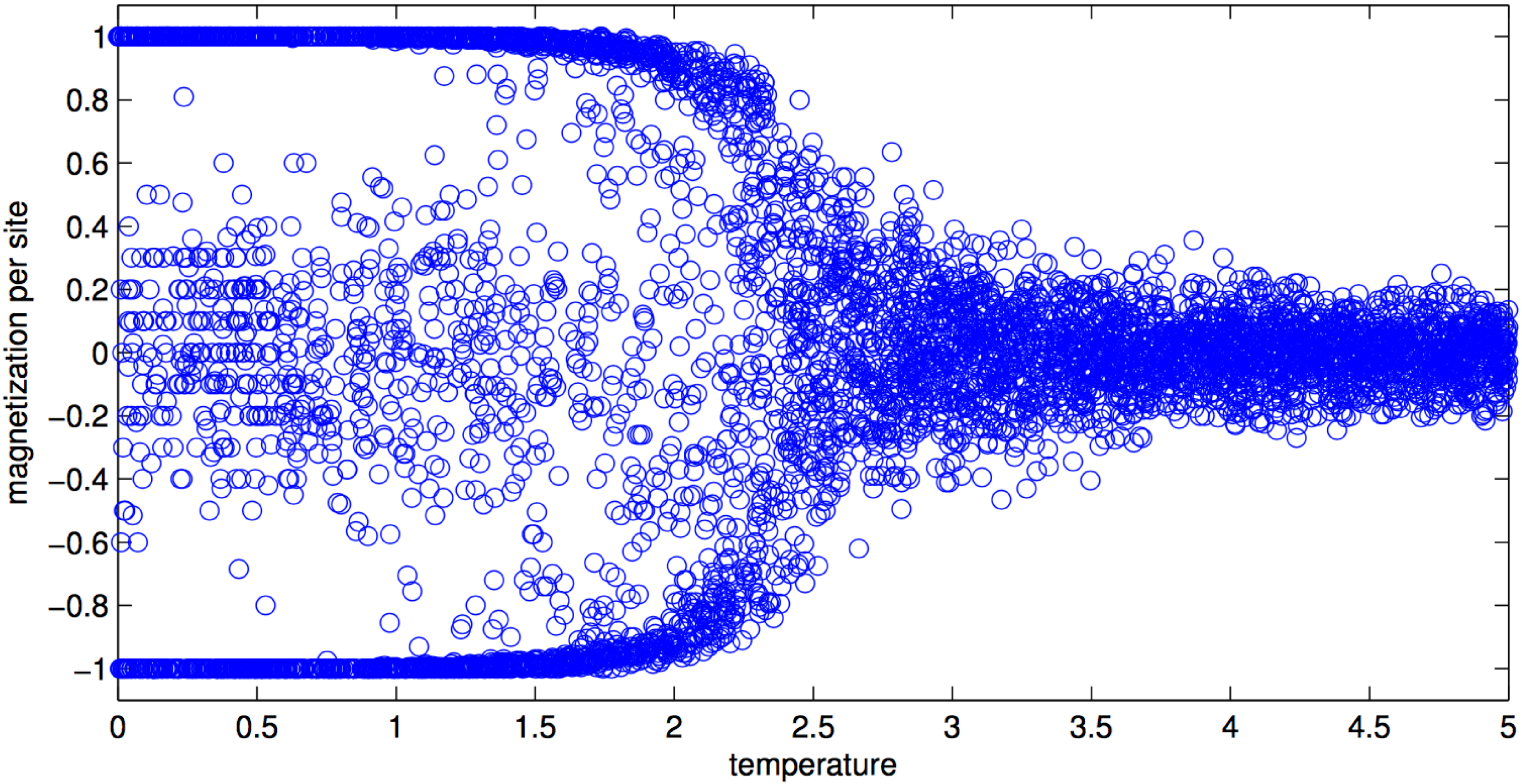
We can model this by randomly proposing that a particle moves 5 steps. The probability is 0.5 of accepting this move.

In other words we say the particle only has a 50% chance of having enough energy to make the move.

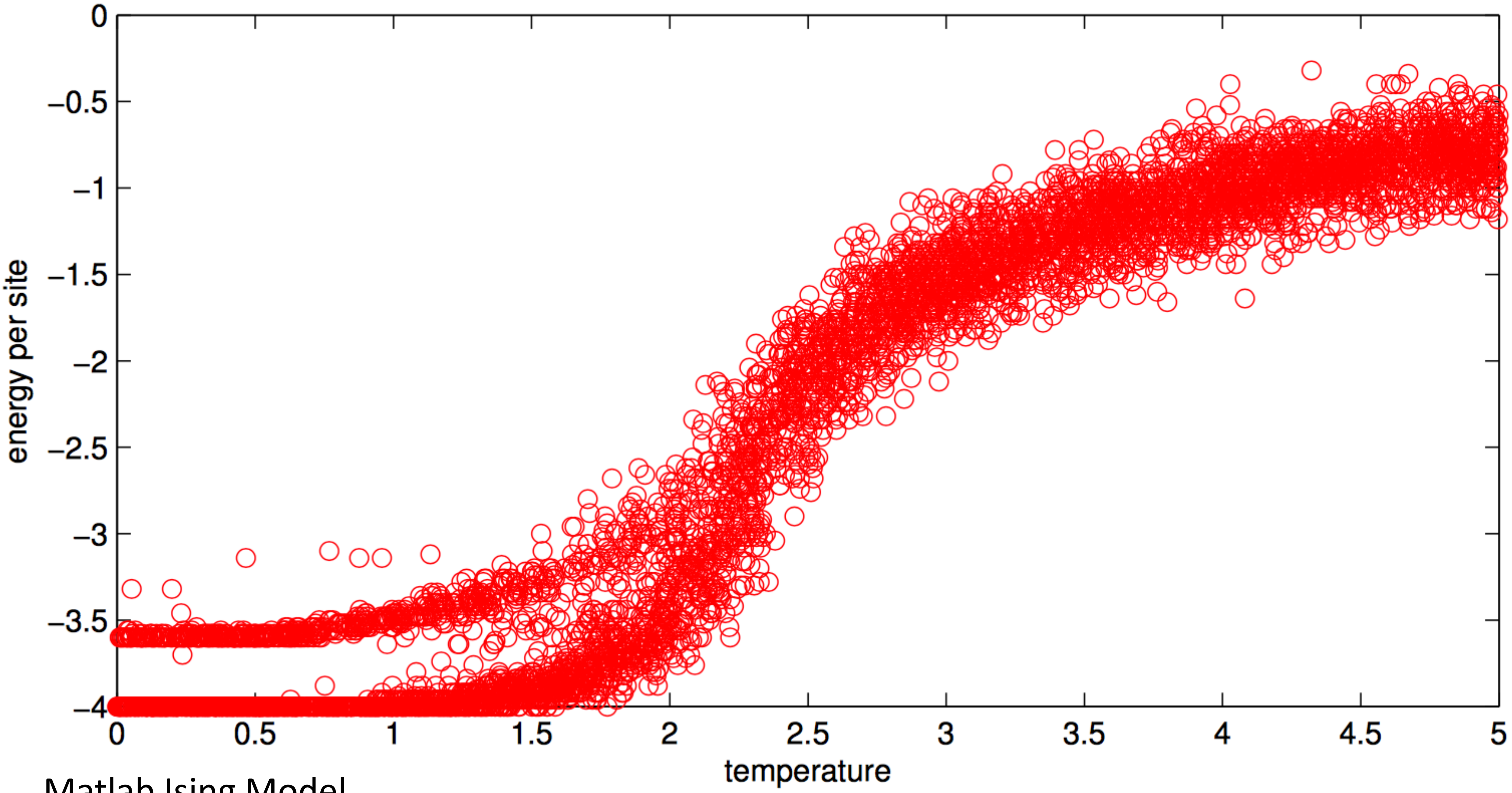
# Ising Model



1.10
1.05
1.00 $\frac{T}{T_c}$
0.95
0.90
1.0
0.5
0.0
−0.5
−1.0
0
400
800
1200
1600
2000
$m$
$E/E_0$
$G(1)$




Matlab Ising Model

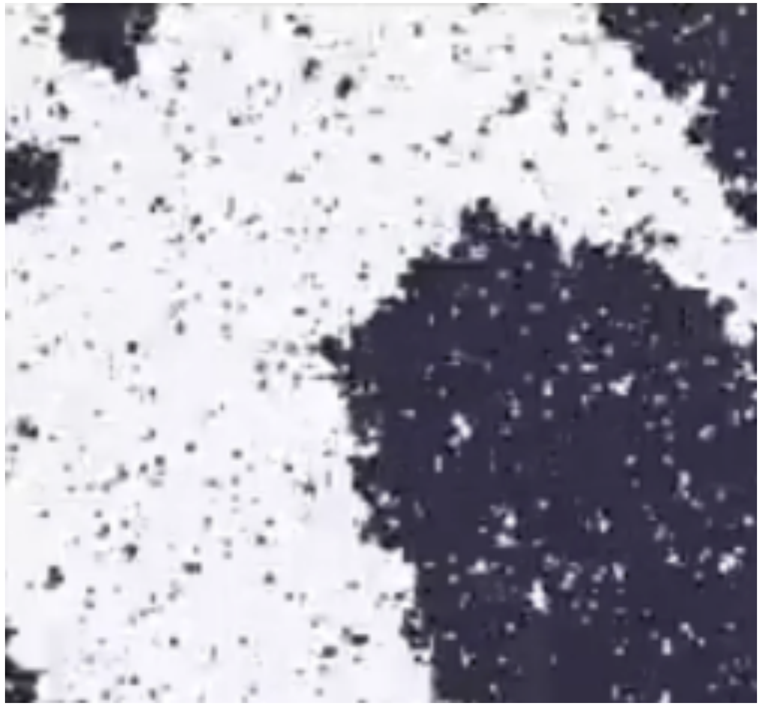



Matlab Ising Model

# Self-organised Criticality

- We can describe systems like this in terms of the correlations. How much do states influence each other.

The particles do not influence
Their neighbourhood much



Particle neighbours are correlated,
short correlation lengths.

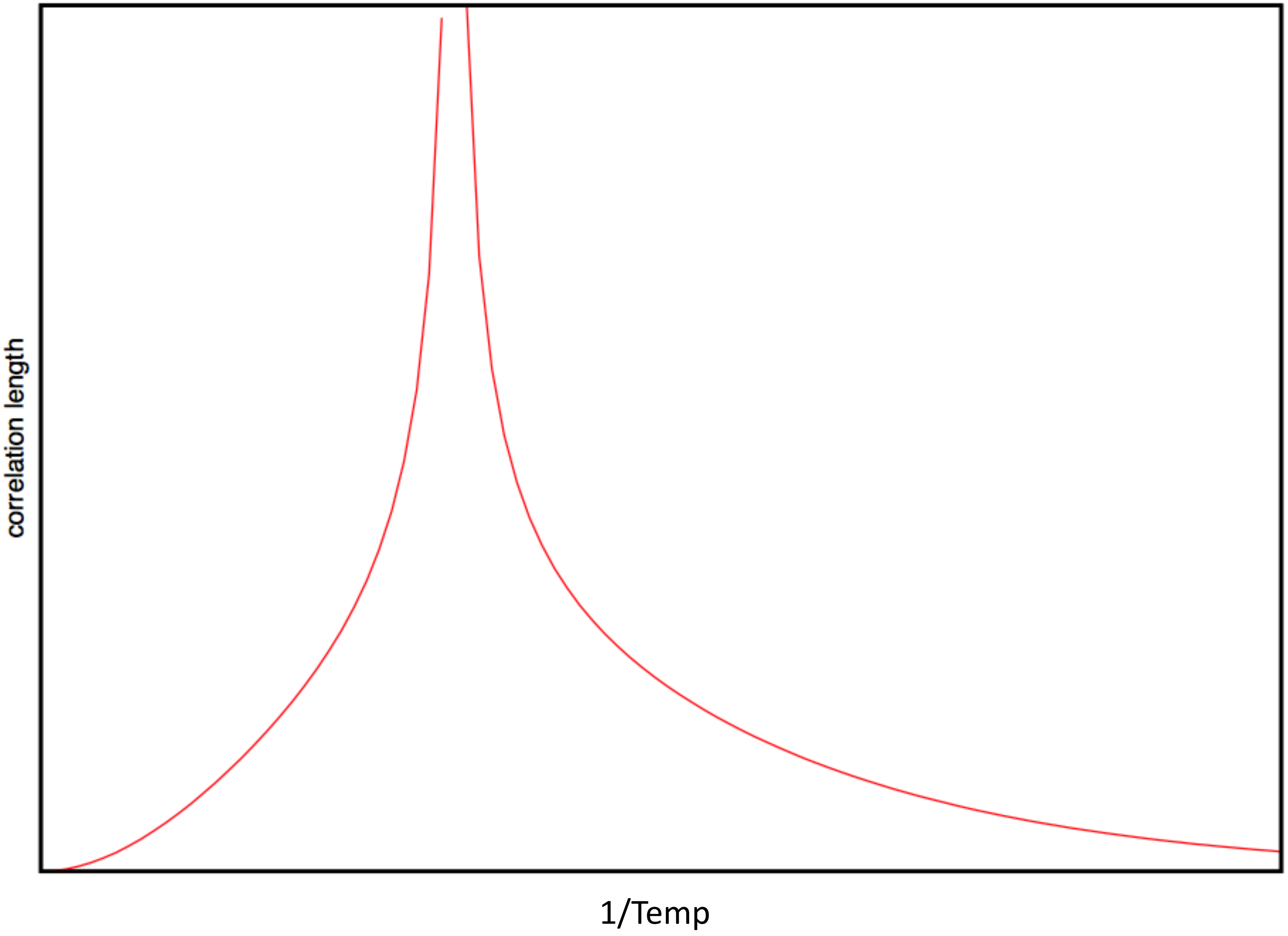
The correlation length is a measure of the characteristic scale of the system.

This is the expected correlation of the spins between all pairs of cells.

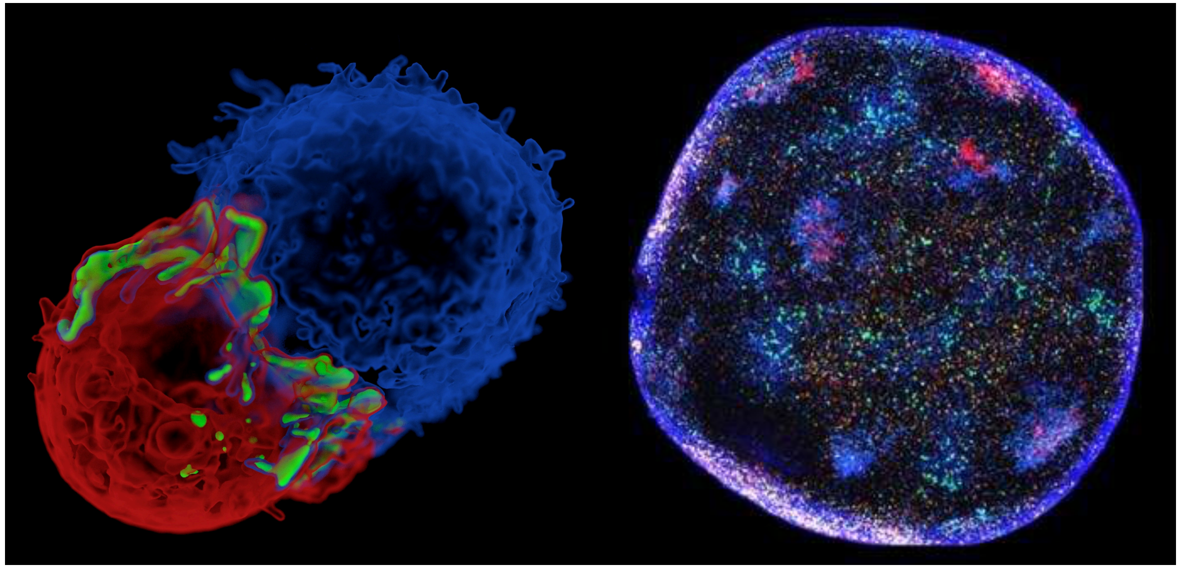
At the phase transition the correlation length becomes undefined.

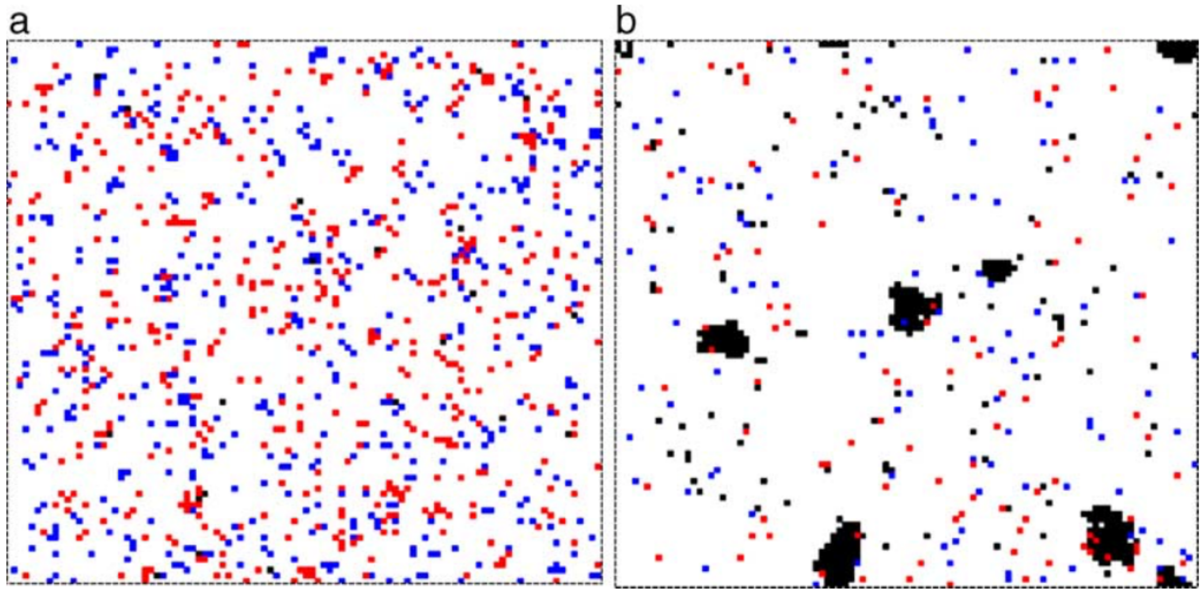
A correlation length of zero means there is no correlation.

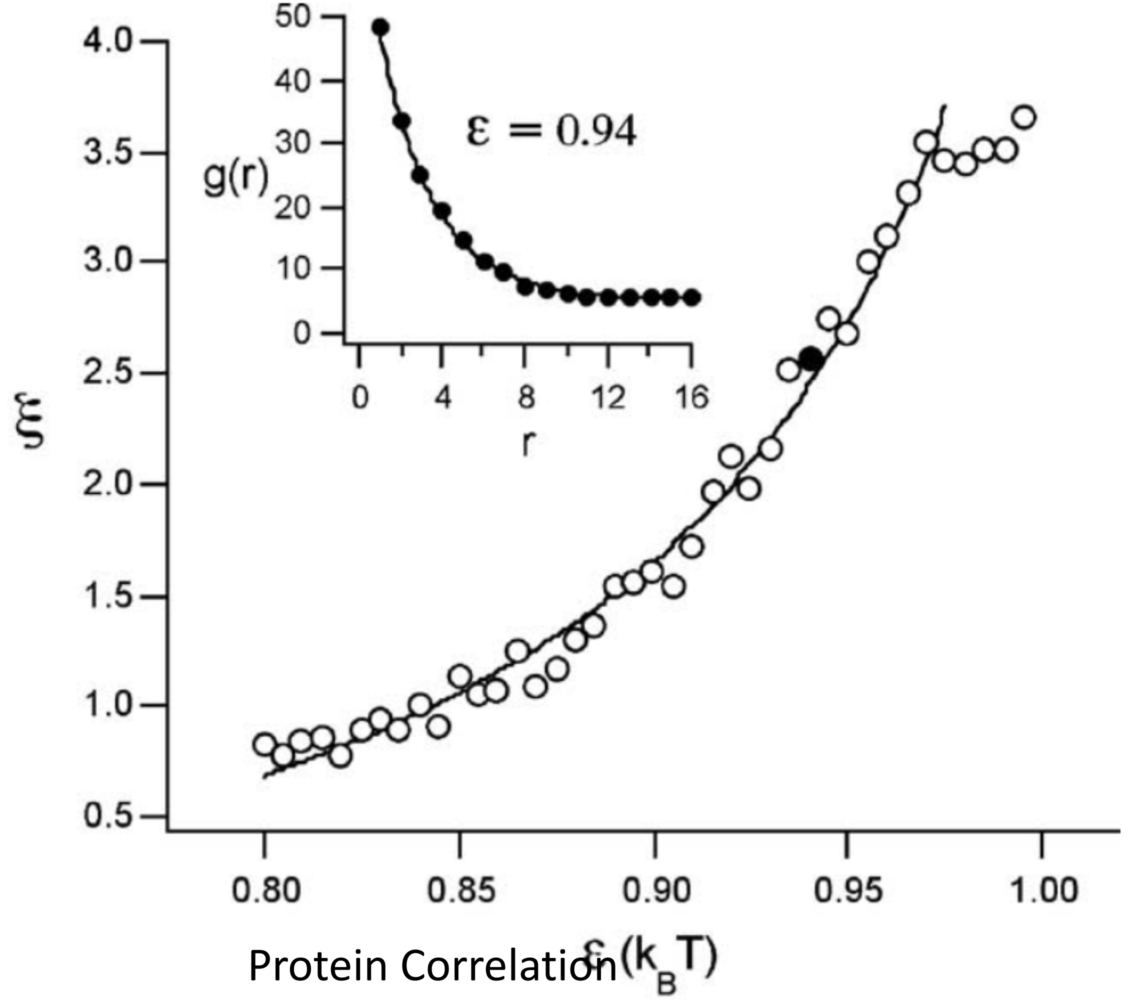
A short correlation length means distant cells are weakly correlated.



# Phase Transitions: Immunology






*Receptor aggregation by intermembrane interactions: A Monte Carlo study* G. Matthew Fricke; James L. Thomas **Biophysical Chemistry** Volume 119, Issue 2, 20 Jan 2006; Pages 205-211.

# Phase Transitions

- This is a key idea in Complex Systems
- When a system transitions from one phase to another (as in the Ising model) they behave in a very special way
- At this transition point the system is said to be *critical*
- Criticality is typically reached in response to tuning some external parameter (temperature in the case of the Ising model)

# Self-Organized Criticality

- Can Complex Adaptive Systems adapt their own parameters so the system can move towards critical states on its own?

# Self-Organized Criticality

- Can Complex Adaptive Systems adapt their own parameters so the system can move towards critical states on its own?

- Yes…
- Originally identified in P. Bak, C. Tang and K. Wiesenfeld, Physical Review Letters 59, 381 (1987).

# Going from fixed points to complexity to chaos are phase transitions

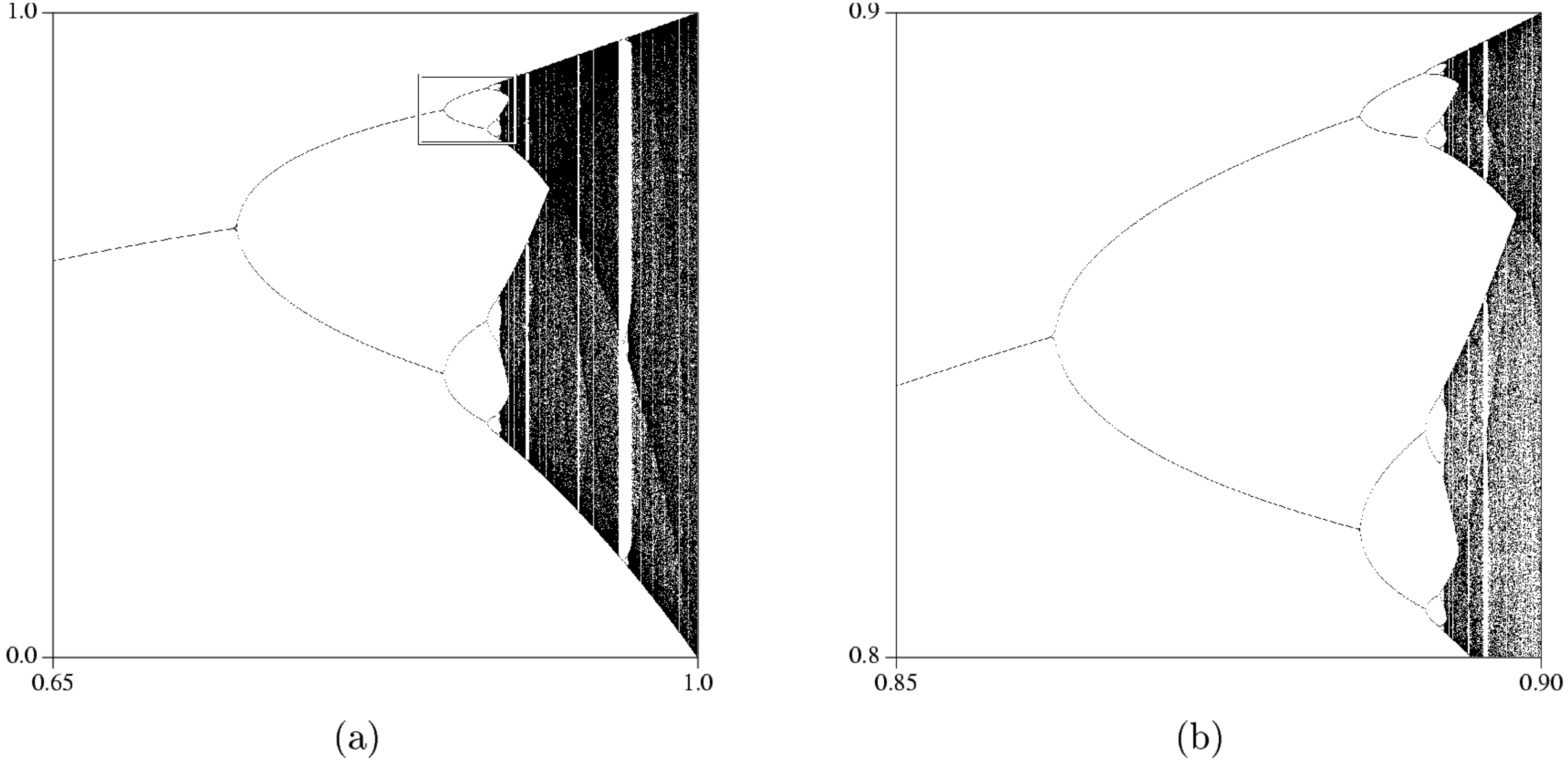
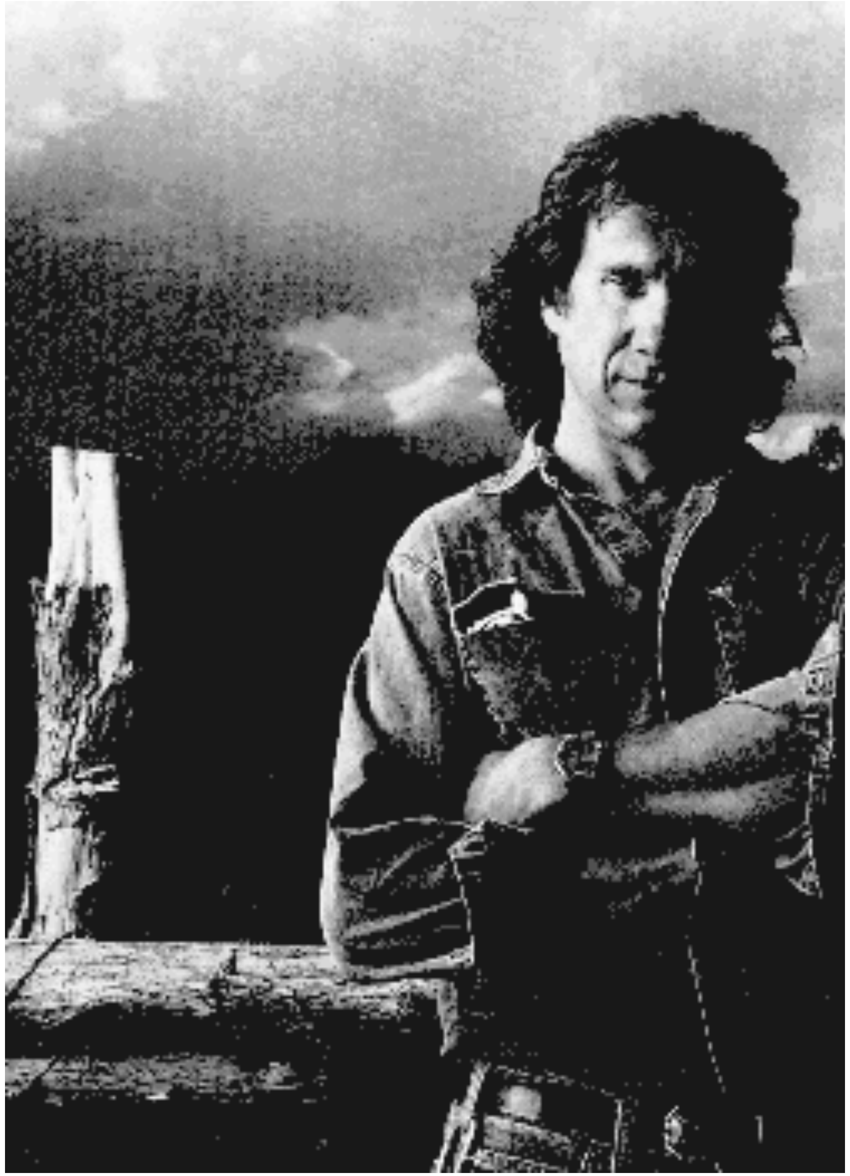



**Figure 10.7** Bifurcation diagrams for the logistic map: (a) This image has values of $r$ such that fixed points, limit cycles, and chaos are all visible. (b) This image shows the detail of the boxed section of (a).

# Self-Organized Criticality

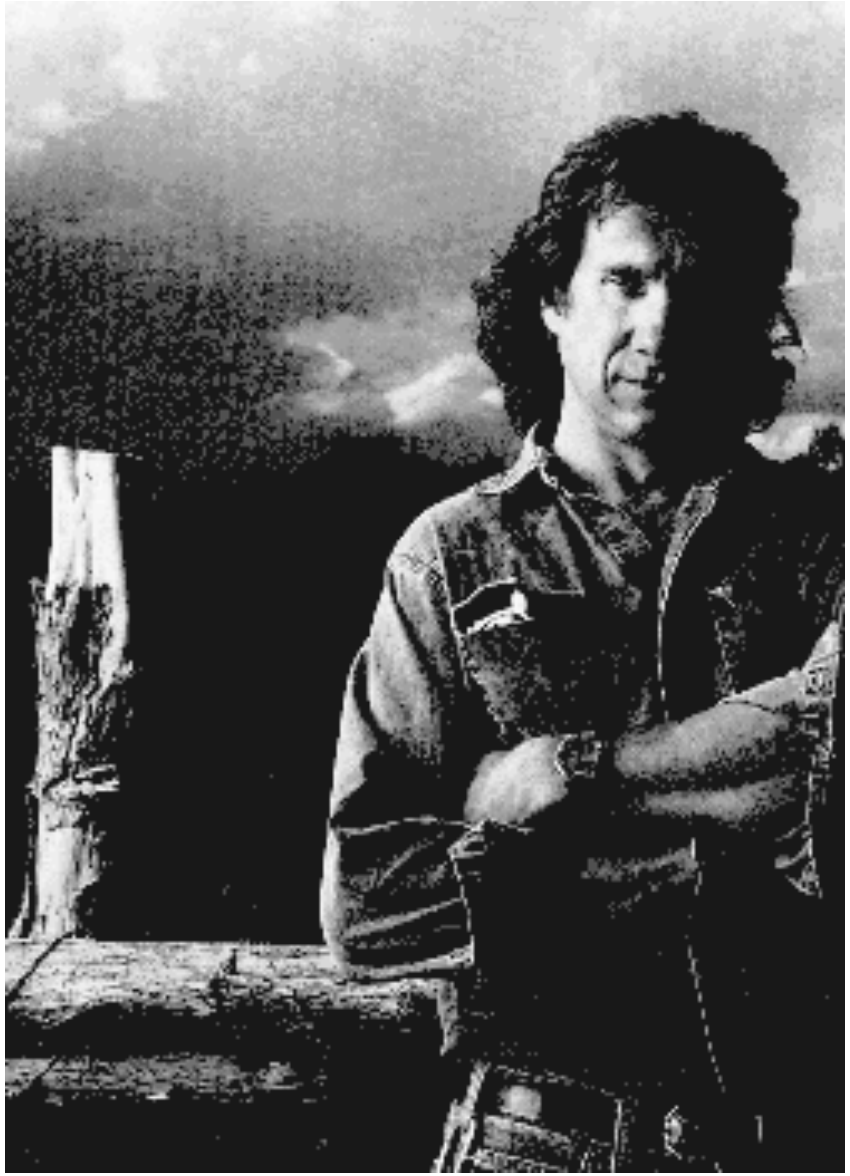
- A recurring observation complex systems is that many systems exist in the phase transition between order and chaos
- For biological systems this is described as “life at the edge of chaos”
- In computer science we refer to ”computation at the edge of chaos”
- Recall Crutchfield describing how systems tendt o balance themselves between order and chaos.



Christopher Langton,
Complex Systems Group, Los Alamos Labs
Santa Fe Institute

# Self-Organized Criticality

- Langton argued that the ability to carry out computations (information storage, transmission, and modification) occur at the phase transition between order and chaos*
- Builds on Wolfram's CA classifications.
- Recall that Class IV CAs can support universal computation.



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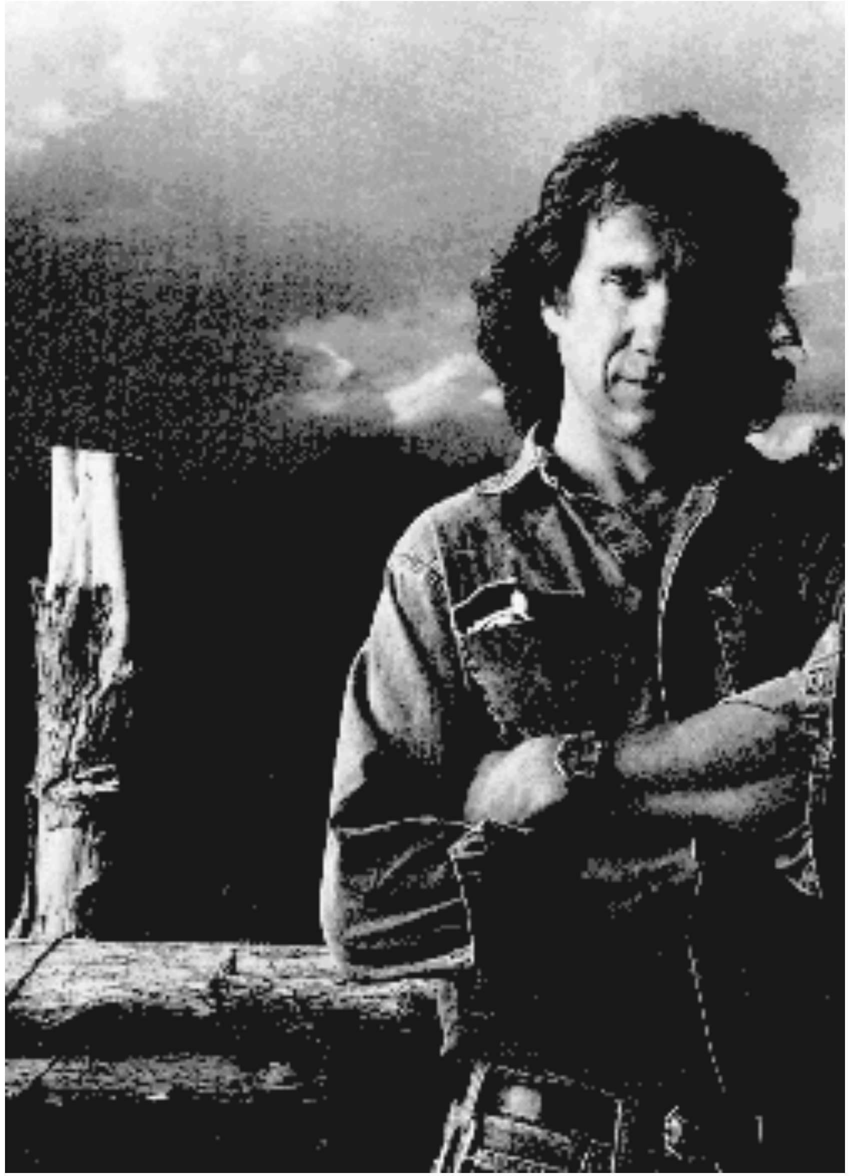
*Langton, Chris G. "Computation at the edge of chaos: phase transitions and emergent computation." *Physica D: Nonlinear Phenomena* 42.1-3 (1990): 12-37.

# Self-Organized Criticality

- Langton showed that the CA rules that support computation tend to exist in the regions between chaos.



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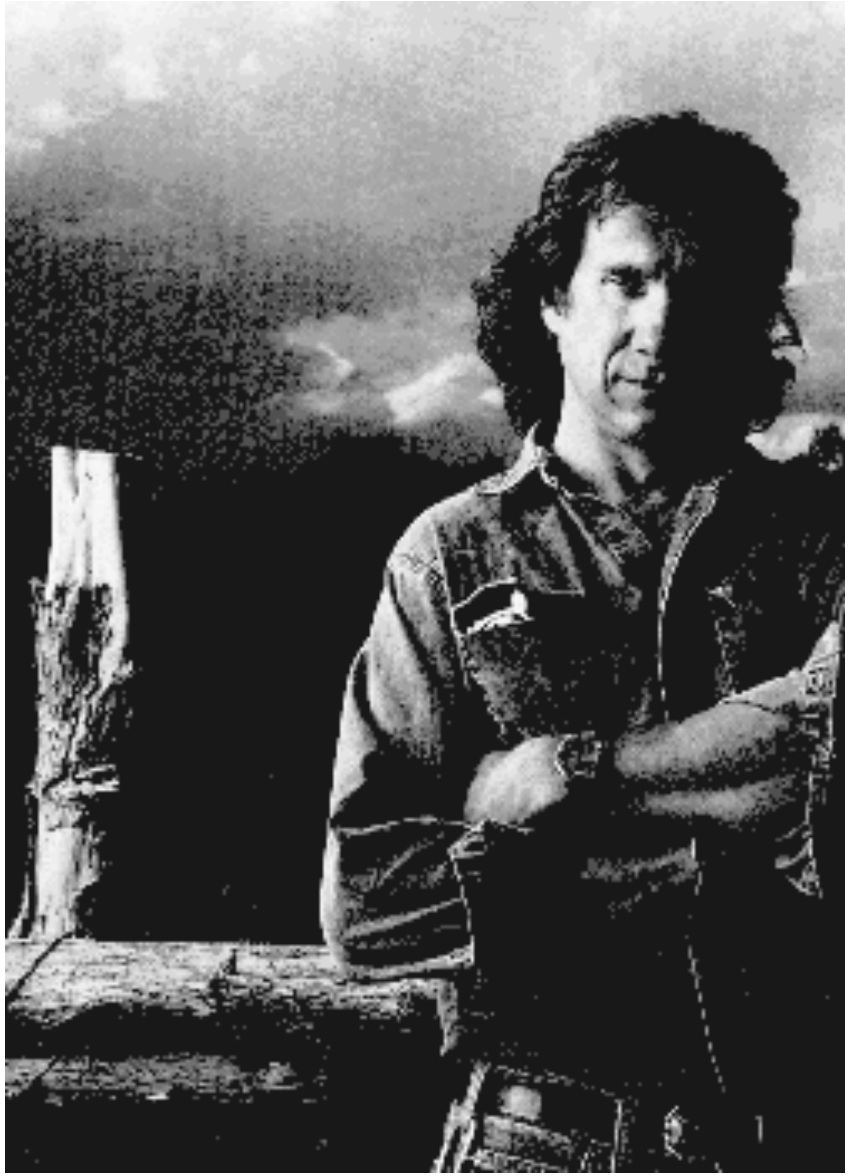
# Self-Organized Criticality

- Langton showed that the CA rules that support computation tend to exist in the regions between chaos.
- We can think of computation, complexity, and living systems as being very long transients.



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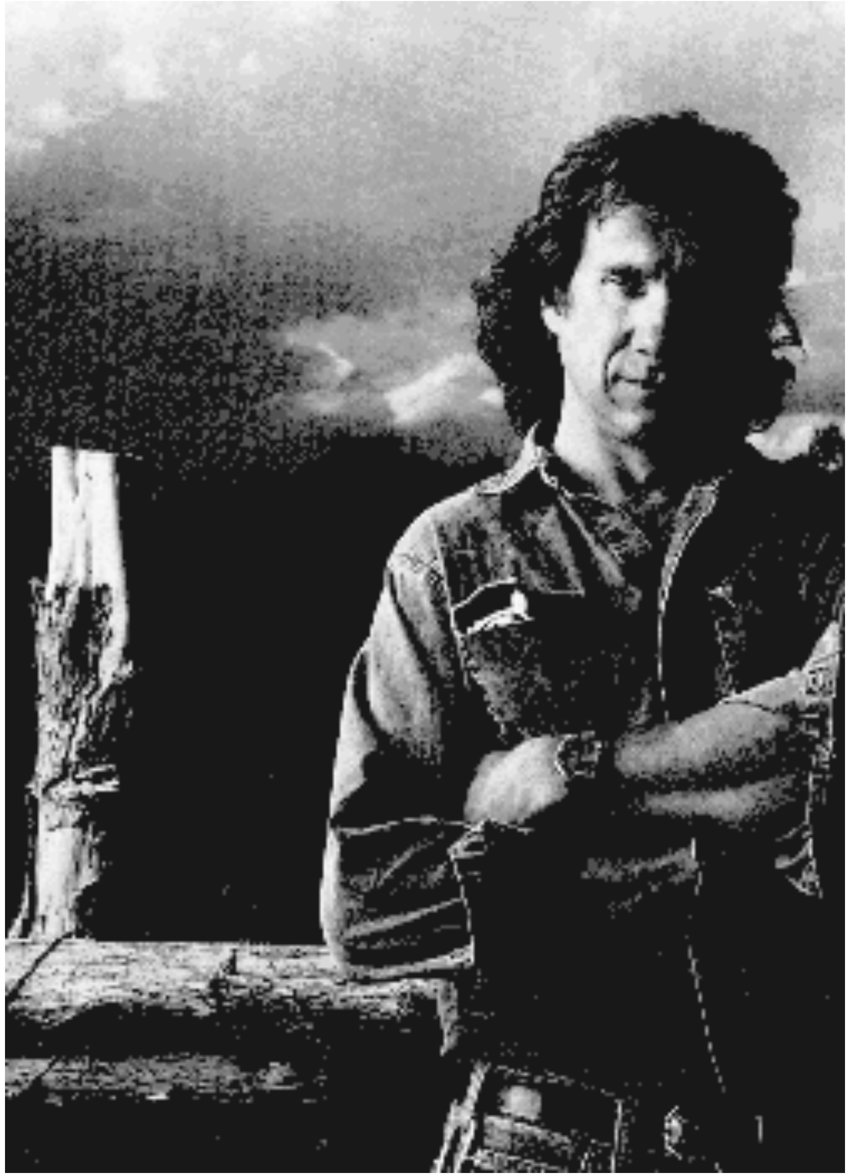
# Langton's λ-parameter

- Wolfram's CA classifications are qualitative and come from inspection of the space-time progression of the CA.
- Langton introduced the λ-parameter as a quantitative measure of complexity.



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# Langton’s λ-parameter

- The lambda parameter of a CA is a number between 0 and 1.
- If lambda is 0, then all cells die immediately, since every rule leads to death.
- If lambda is 1, then any cell that has at least one living neighbor will stay alive in the next generation and, in fact, forever.
- Values of lambda close to zero give CA's in the ordered realm.
- Values close to 1 give CA's in the chaotic realm.
- The edge of chaos is somewhere in between.

*Langton, Chris G. "Computation at the edge of chaos: phase transitions and emergent computation." *Physica D: Nonlinear Phenomena* 42.1-3 (1990): 12-37.

# Langton's λ-parameter

- λ is a statistic of the output states in the CA lookup table, defined as the fraction of non-quiescent states in this table.
- The quiescent state is an arbitrarily chosen state (for example 0 in a CA with possible states {0,1})
- λ is the fraction of rules in the CA that do not lead to quiescence (death)

$$\lambda = \frac{K^n - n}{K^n}$$

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# Langton's λ-parameter

Each finite automaton consists of a finite set of *cell states* Σ, a finite *input alphabet* α, and a *transition function* Δ, which is a mapping from the set of neighborhood states to the set of cell states.

$$\Delta : \Sigma^N \to \Sigma$$

Where $N$ is the size of the rules neighbourhood.

$\Sigma^N$ is the set of possible inputs to each rule.

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# Langton's λ-parameter

There are $|\Sigma^N|^{|\Sigma|^N}$ possible transition functions.

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# Langton's λ-parameter

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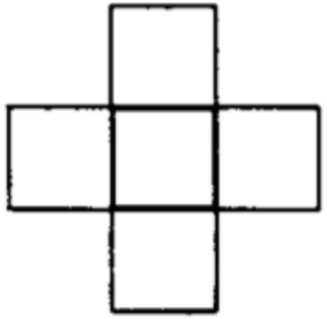
e.g.



five cell neighborhood

8 states per cell

$8^{8^5} = 10^{30000}$ possible transition functions

Where $N$ is the size of the rules neighbourhood.

$\Sigma^N$ is the set of possible inputs to each rule.

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# Langton's λ-parameter

The $\lambda$ parameter is defined as follows.

We pick an arbitrary state to be the quiescence state $s_q$.

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Let there be $n$ transitions to this special quiescent state in a transition function $\Delta$

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Let there be $n$ transitions to this special quiescent state in a transition function $\Delta$

Let the remaining $|\Sigma^N| - n$ transitions in $\Delta$ be filled randomly from the other $|\Sigma| - 1$ states.

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# Langton’s λ-parameter

$$\lambda = \frac{|\Sigma^N| - n}{|\Sigma^N|}$$

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$$\lambda = \frac{|\Sigma^N| - n}{|\Sigma^N|}$$

Which is just the fraction of inputs that cause the CA rule to explore the state space

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# Langton's λ-parameter

$$\lambda = \frac{|\Sigma^N| - n}{|\Sigma^N|}$$

Which is just the fraction of inputs that cause the CA rule to explore the state space

If it explores too much it is chaotic, too little and it is ordered.

*Langton, Chris G. "Computation at the edge of chaos: phase transitions and emergent computation." *Physica D: Nonlinear Phenomena* 42.1-3 (1990): 12-37.

Game of life has $\lambda = 0.273$

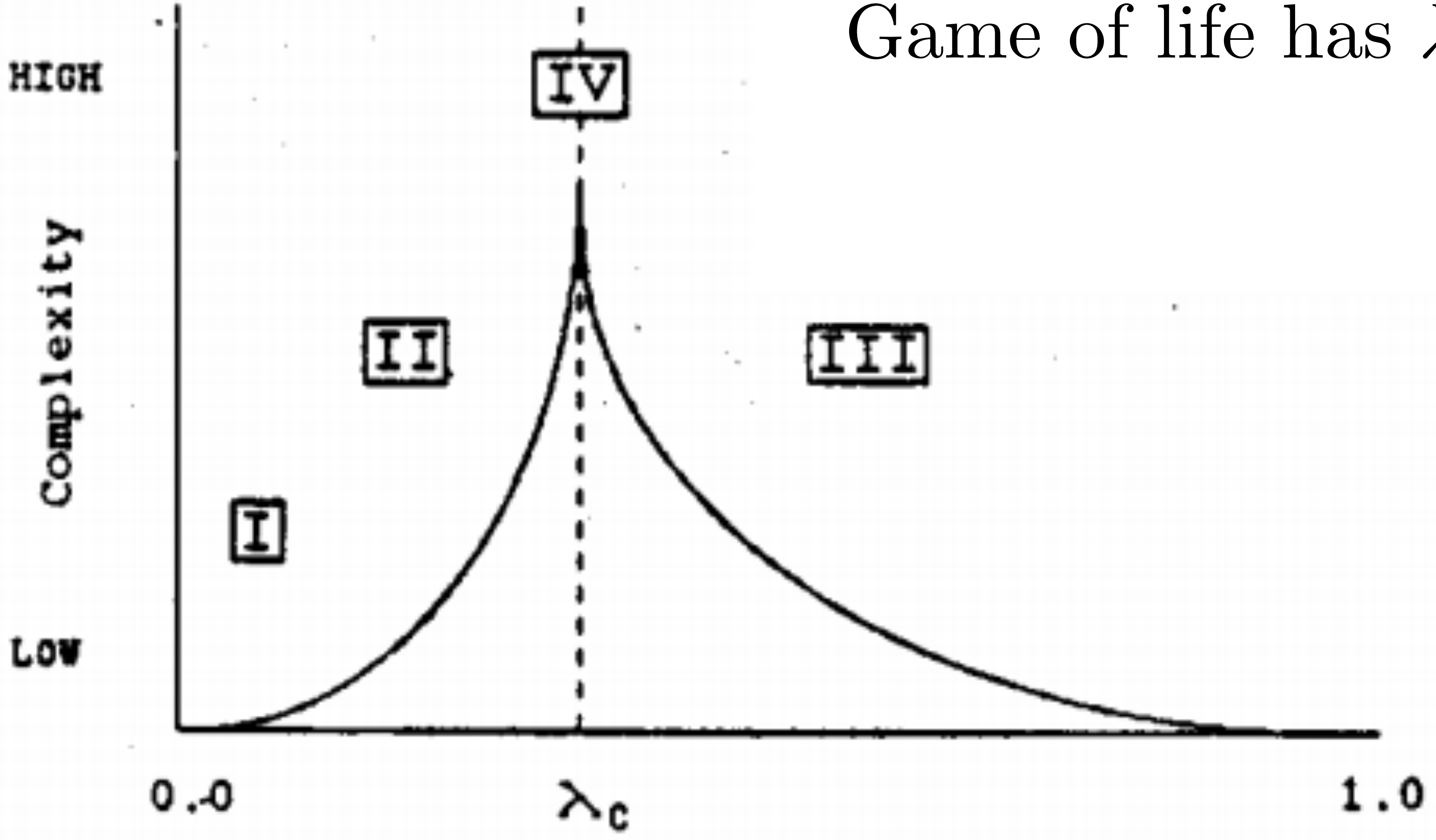



Fig. 16. Location of the Wolfram classes in λ space.

Recall that entropy is a measure of order vs randomness.

(Note: chaos is NOT random it is deterministic, but it *looks* random). It looks random to entropy measures as well at qualitatively.

Near the 0.5 the information entropy changes character.

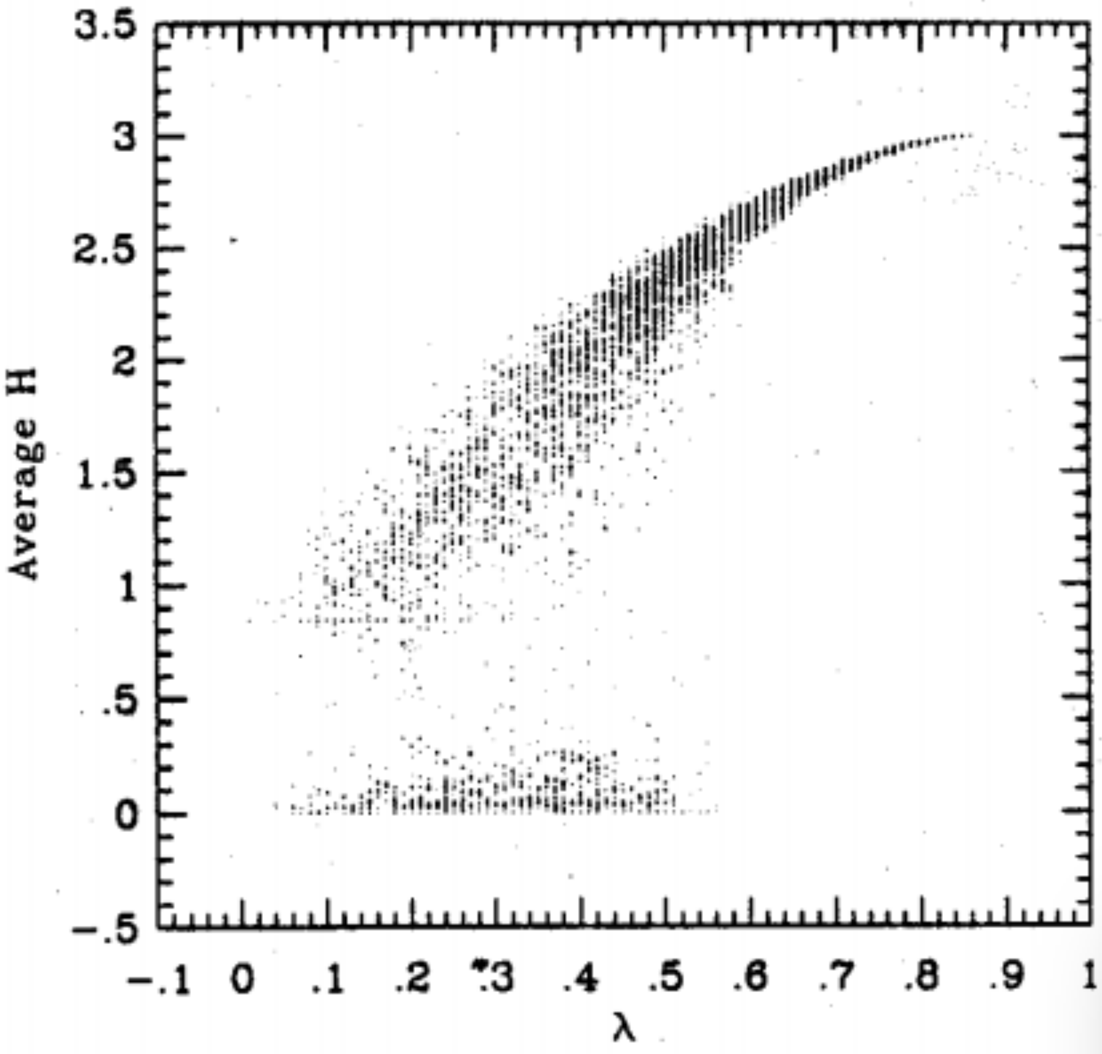



Fig. 6. Average single cell entropy $\overline{H}$ over λ space for approximately 10 000 CA runs. Each point represents a different transition function.

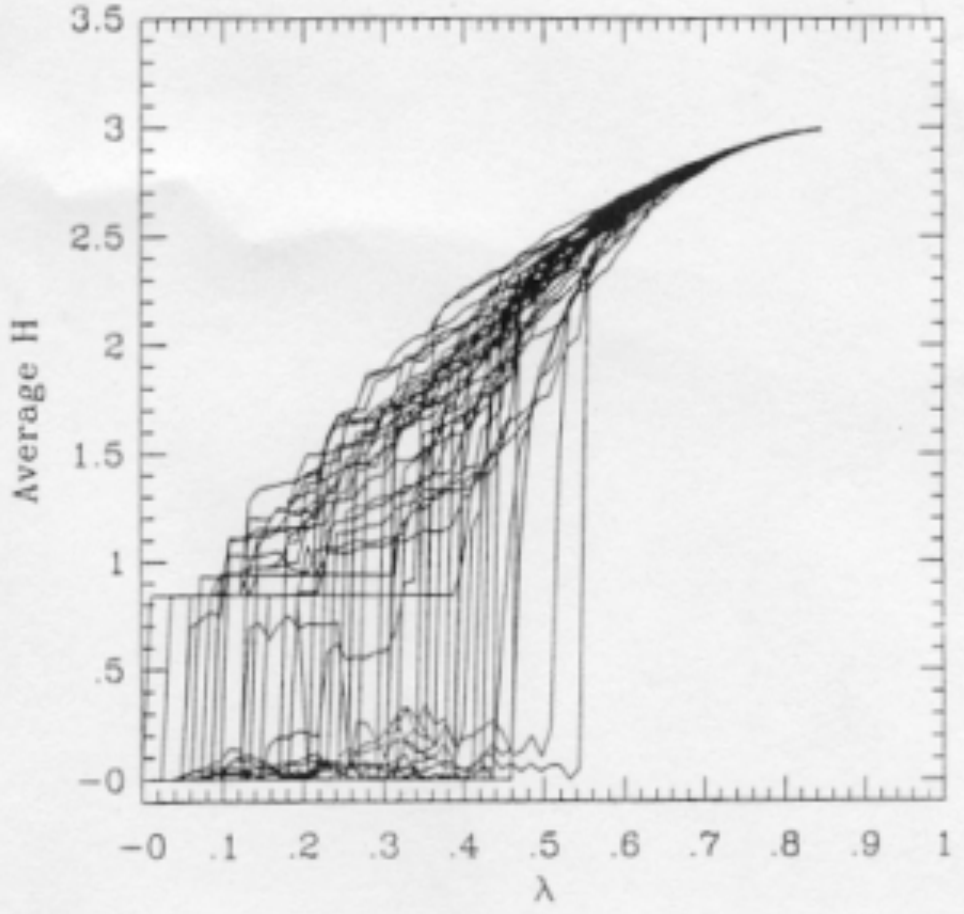



Fig. 8. Superposition of 50 transition events, showing the internal structure of fig. 6.

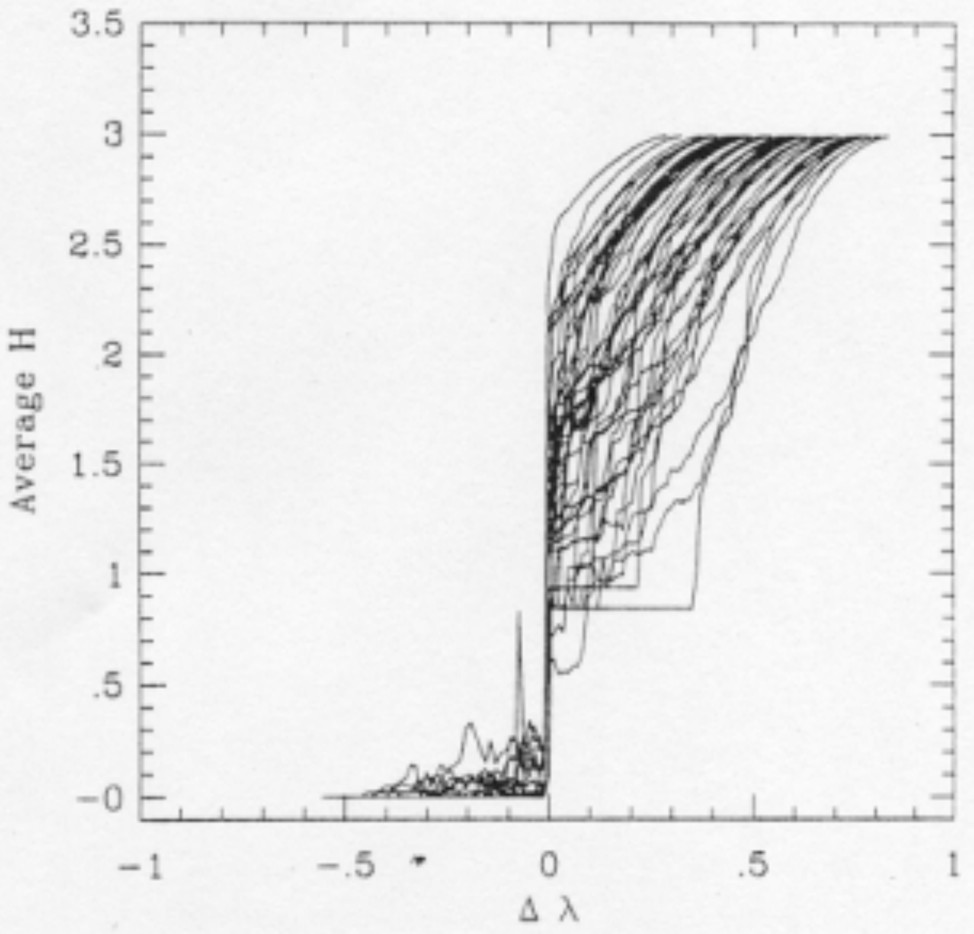



Fig. 10. Superposition of 50 transition events lined up by Δλ. Compare with fig. 8.

Phase transitions match up with Langton's lambda parameter

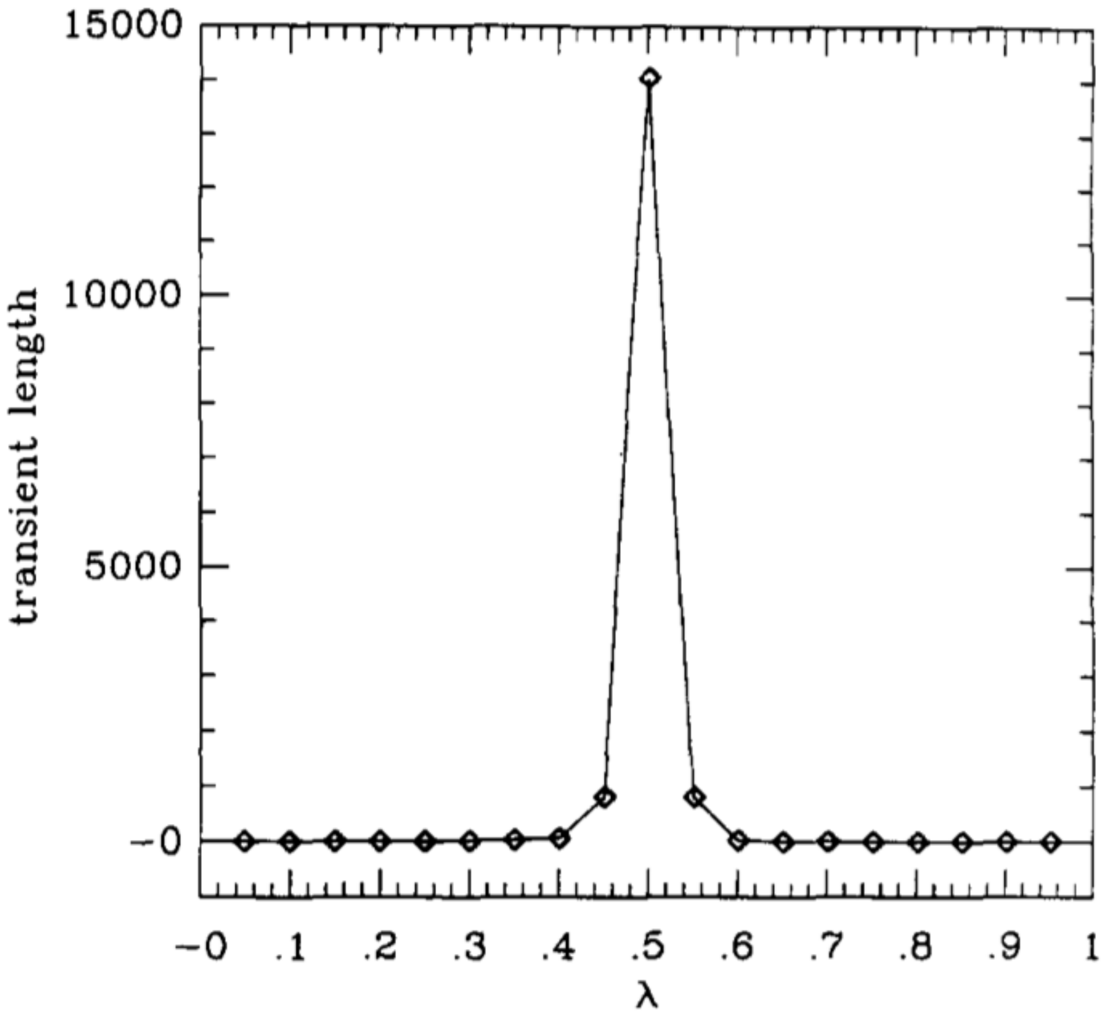
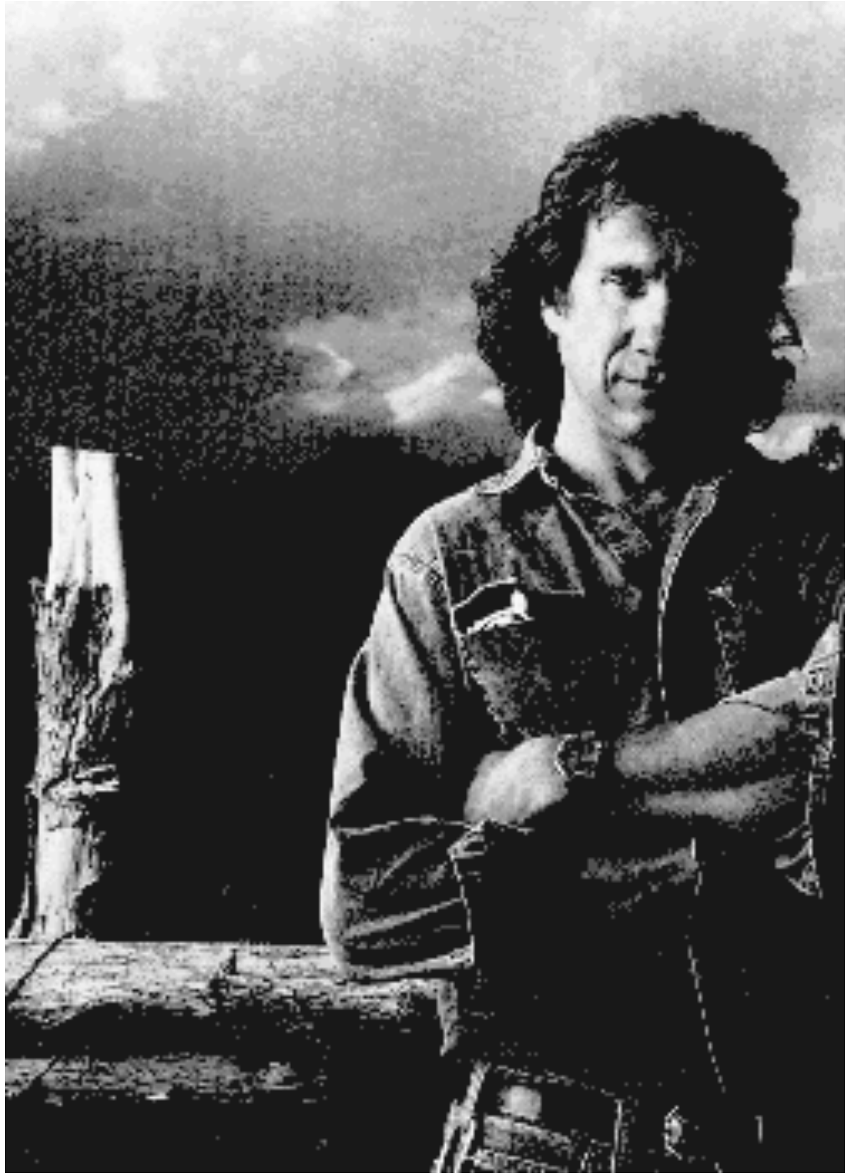



Fig. 3. Average transient length as a function of λ in an array of 128 cells.



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Below .5 is ordered, above 0.5 is chaos.
Transients are a proxy for complexity of computation.

# Life at the Edge of Chaos

Stuart Kauffman argues that evolution tends to push life
To the edge of chaos.

Evolution is a powerful mechanism, but exploring the biochemical landscape seems daunting even with the enormous number of trials conducted over the past few billion years.

Kauffman's work has focused on the underlying structures that evolution is able to exploit.

The edge of chaos is one of the places that evolution is able to exploit in order to more easily find solutions.

We can think of evolution as using the edge of chaos as being a possible GA *building block*.

Stuart Kauffman, *At Home in the Universe: The Search for Laws of Self-organization and Complexity*. Oxford University Press, 1995.

**Second-Order Phase Transitions** Phase transitions occur in many physical systems when the number of components diverges, viz "macroscopic" systems. Every phase has characteristic properties. The key property, which distinguishes one phase from another, is denoted the "order parameter". Mathematically one can classify the type of ordering according to the symmetry of the ordering breaks.

> *The Order Parameter.* In a continuous or "second-order" phase transition the high-temperature phase has a higher symmetry than the low-temperature phase and the degree of symmetry breaking can be characterized by an order parameter $\phi$.

Note that all matter is disordered at high enough temperatures and ordered phases occur at low to moderate temperatures in physical systems.

# Self-organised Criticality

**Scale-Invariance and Self-Similarity** If a control parameter, often the temperature, of a physical system is tuned such that it sits exactly at the point of a phase transition, the system is said to be critical. At this point there are no characteristic length scales.

> *Scale Invariance.* If a measurable quantity, like the correlation function, decays like a power of the distance $\sim (1/r)^\delta$, with a critical exponent $\delta$, the system is said to be critical or scale-invariant.

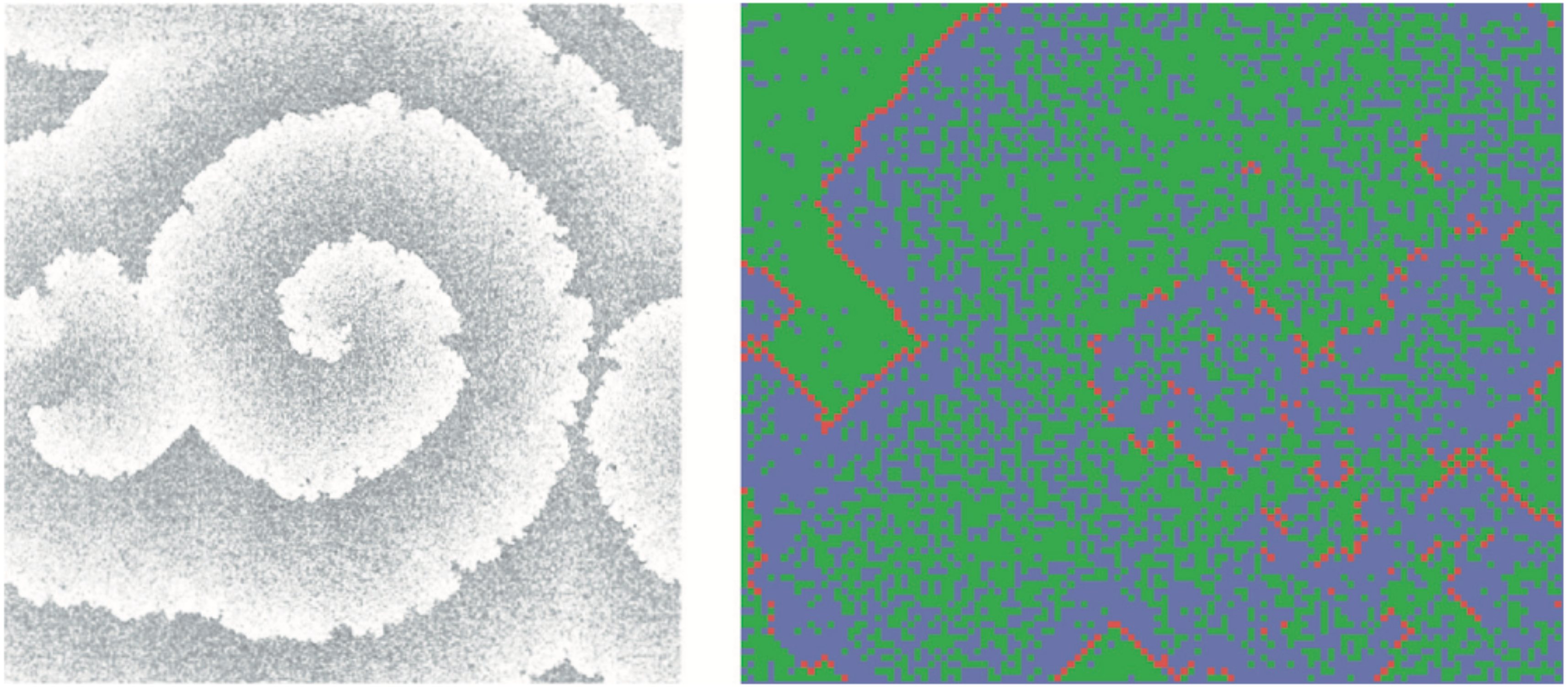
Power laws have no scale; they are self-similar,

$$S(r) = c_0 \left(\frac{r_0}{r}\right)^\delta \equiv c_1 \left(\frac{r_1}{r}\right)^\delta, \qquad c_0 \, r_0^\delta = c_1 \, r_1^\delta \,,$$

# Scale Free

- At the exact point of criticality the correlation lengths between particles in the system become scale free.
- Scale free systems are characterised by power-laws such as the ones Melanie showed.
- Structures within the system reach across all levels at the critical point.

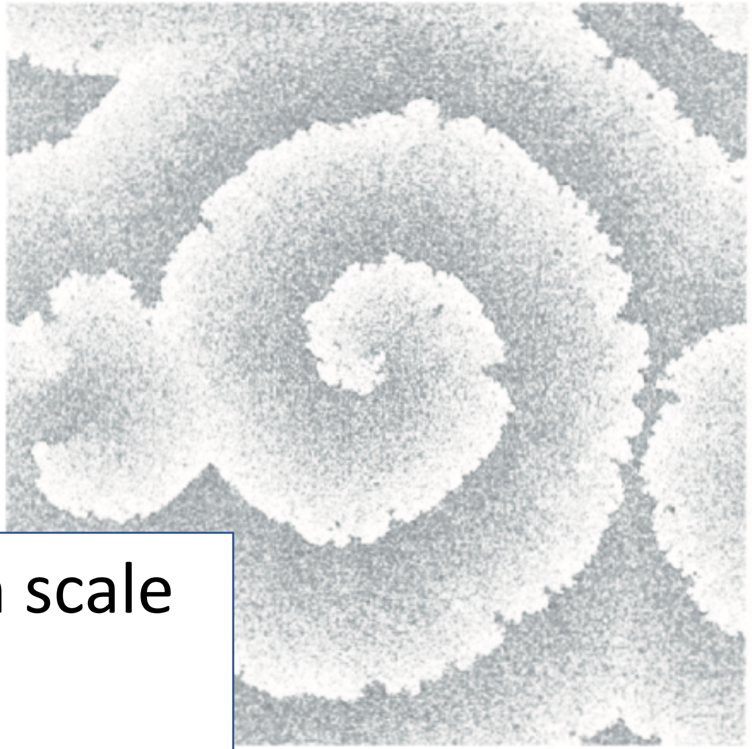
# Forest Fire Model

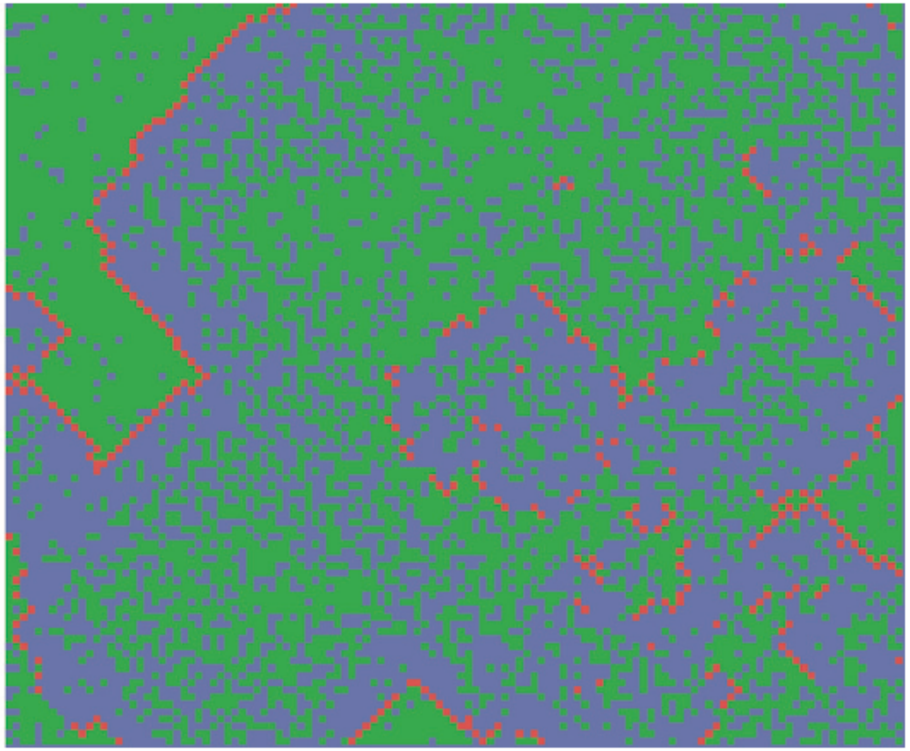


Simulations of the forest fire model. *Left*: Fires burn in characteristic spirals for a growth probability $p = 0.005$ and no lightning, $f = 0$ (from Clar et al. 1996). *Right*: A snapshot of the forest fire model with a growth probability $p = 0.06$ and a lightning probability $f = 0.0001$. Note the characteristic fire fronts with trees in front and ashes behind

CLAR, S., DROSSEL, B., SCHWABL, F. 1996 Forest fires and other examples of self-organized criticality. *Journal of Physics: Condensed Matter* **8**, 6803–6824.

# Forest Fire Model





Characteristic length scale

(Spiral pattern with characteristic time and length scales = 1/p)

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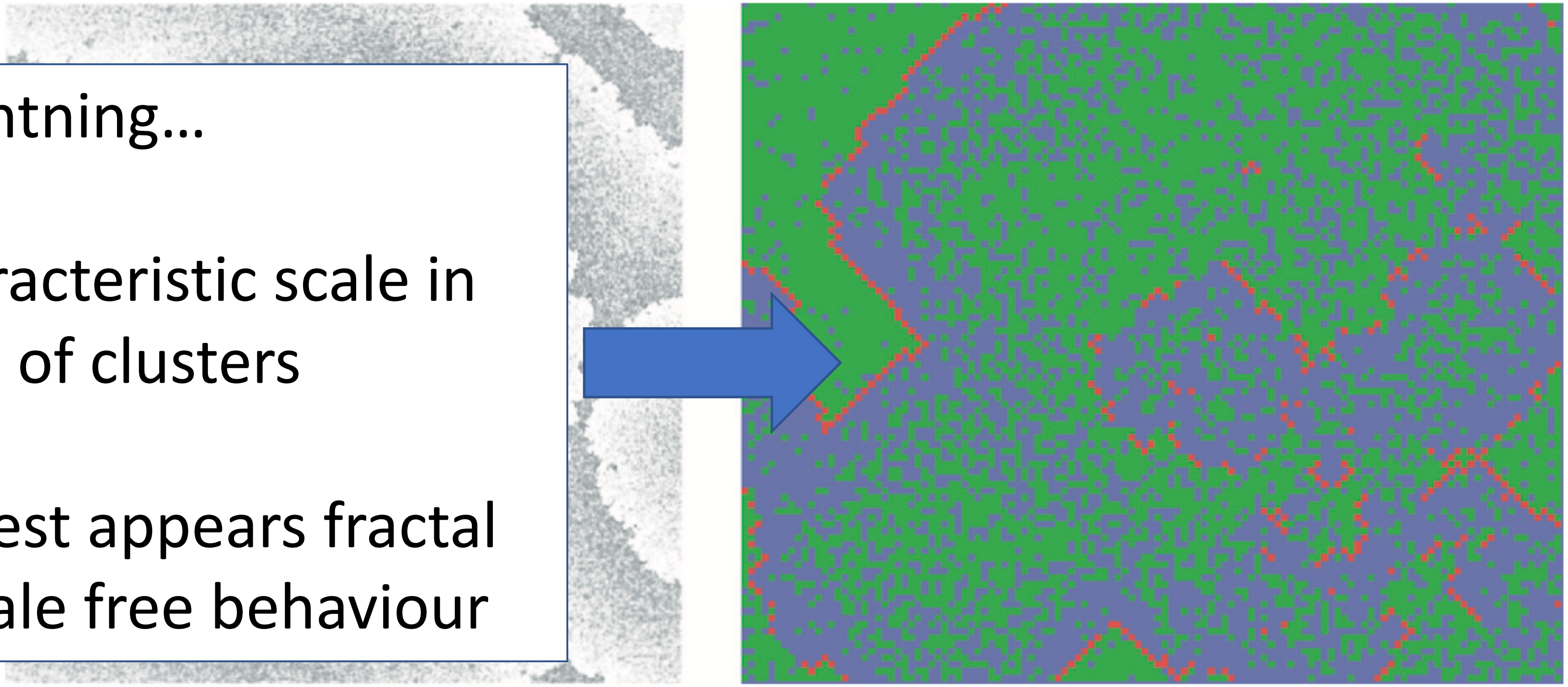
# Forest Fire Model

Add lightning…

No characteristic scale in the size of clusters

The forest appears fractal with scale free behaviour



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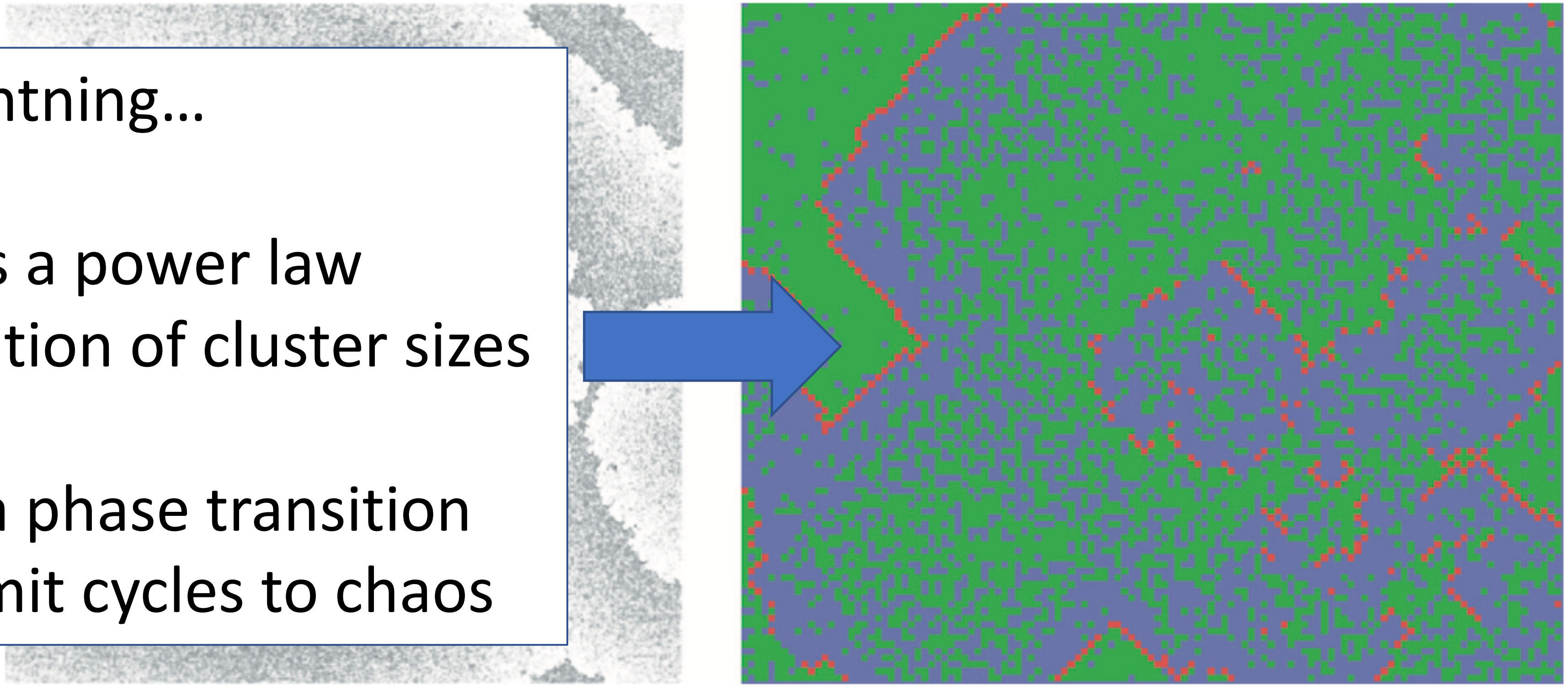
# Forest Fire Model

Add lightning…

There is a power law distribution of cluster sizes

This is a phase transition from limit cycles to chaos



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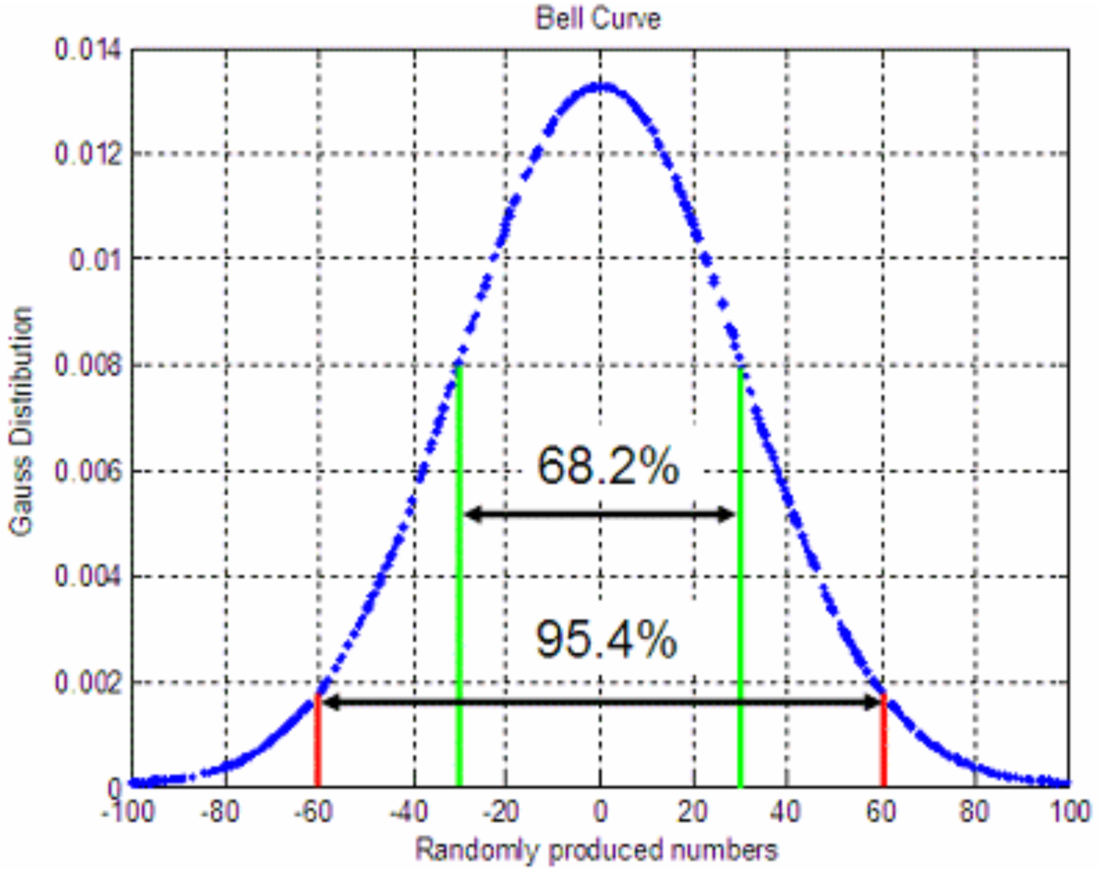
# Self-organised criticality

Gaussian distribution was discovered in the early 19th century by Carl Gauss and Abraham de Moivre
It revolutionised statistics because it explained so many different systems.

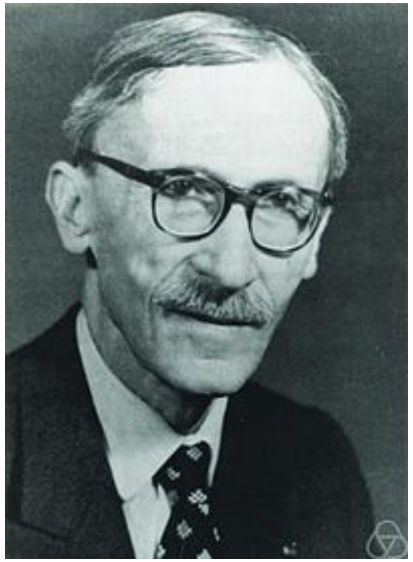
So common it became known as the Normal distribution.

All properties of a system with a characteristic scale converge to this distribution.



# Self-organised criticality

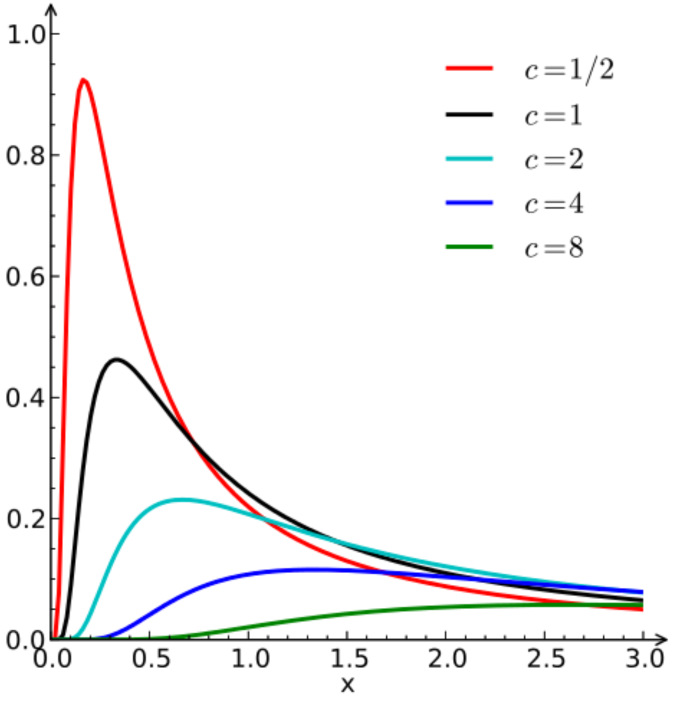
Last century Paul Lévy showed that if we do not have a characteristic scale systems converge to a Lévy distribution (a power law).

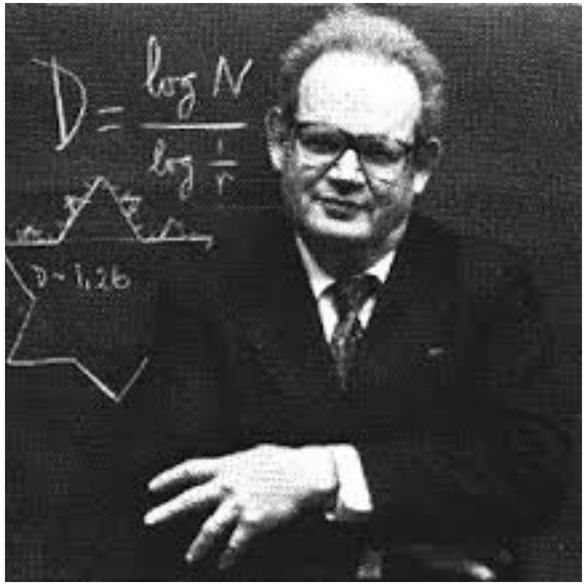


Paul Lévy

Lévy's student Benoit Mandlebrot began the study of fractals.

The discovery that many systems can be described by power-laws had a similar effect to the discovery of the Normal distribution.




enoit Mandlebrot

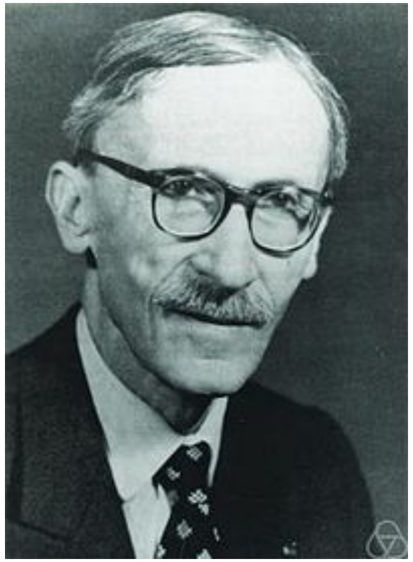
# Self-organised criticality

Many systems can be described by power-laws.

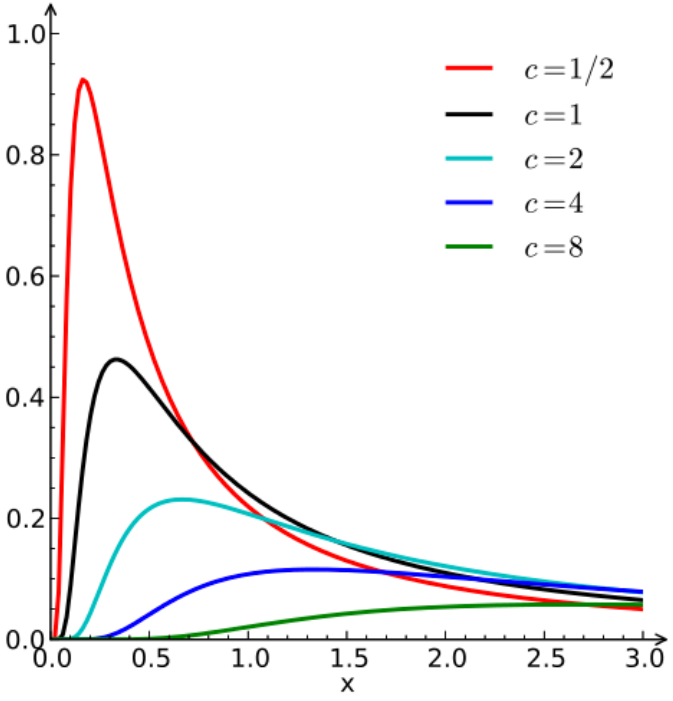
Melanie showed several examples last week:

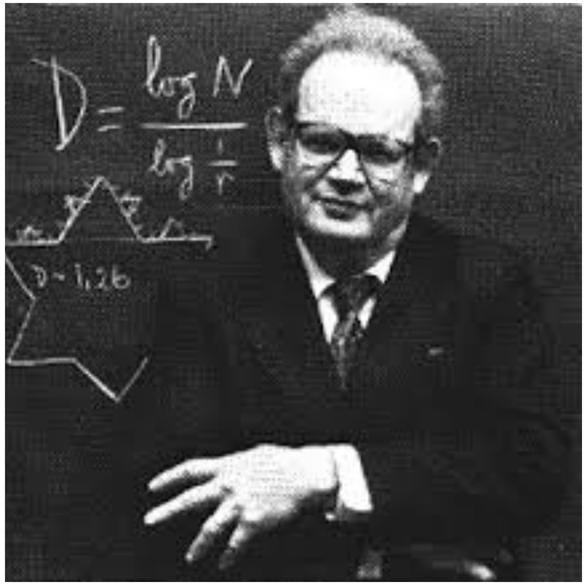
- Biological scaling in metabolism
- Social scaling in cities

We have seen that fractals
Show up in strange attractors
(Chaotic)



Paul Lévy




enoit Mandlebrot

# Self-organised criticality

Lévy walks are a type of search pattern that has been shown to be optimal under many circumstances.

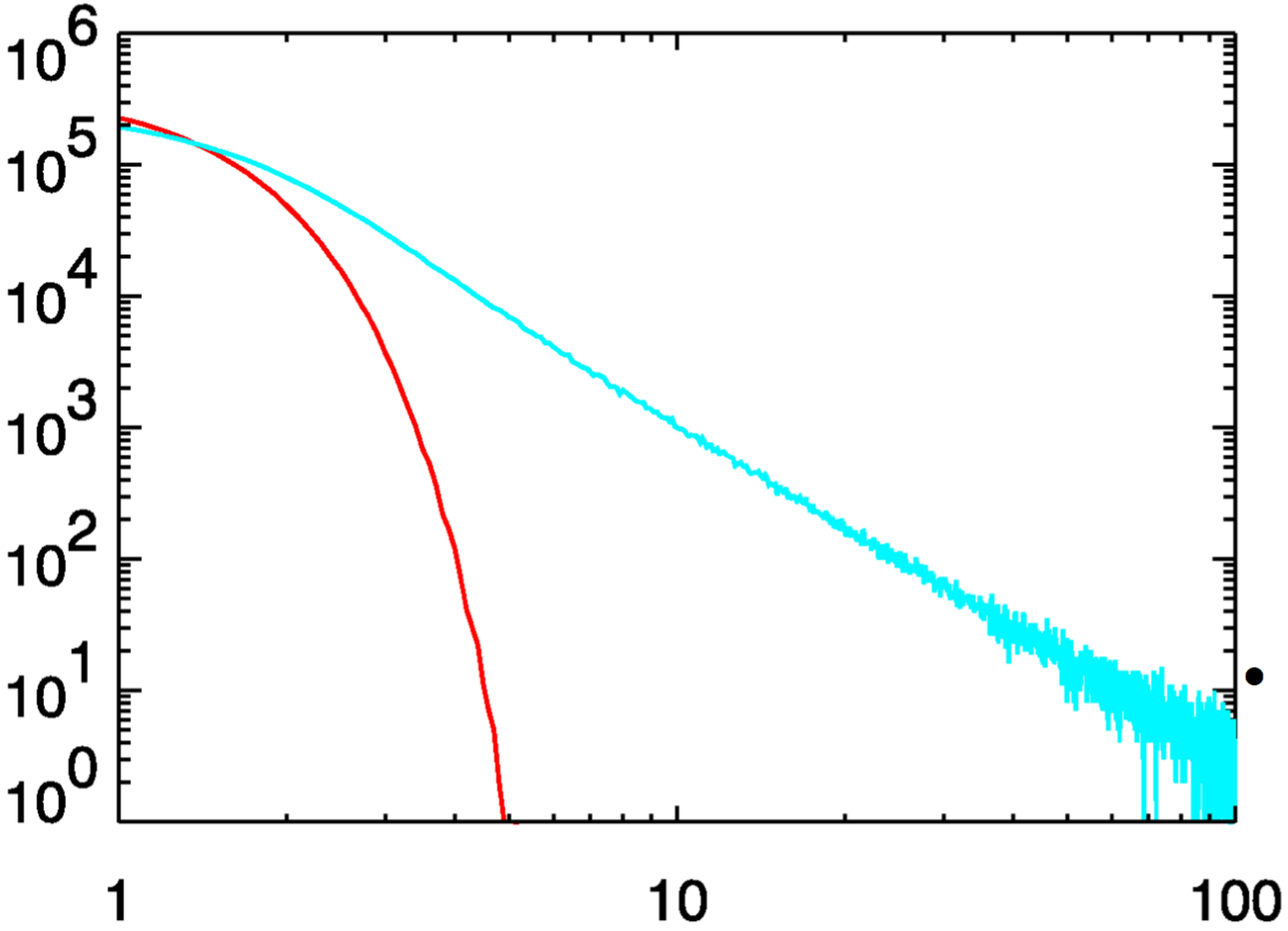
These are scale free search patterns that have been used to model the foraging patterns of numerous species.

Gandi Viswanathn, UNM Postdoc under Nitant Kenkre in the physical dept.

Viswanathan, Gandimohan M., et al. "Optimizing the success of random searches." *Nature* 401.6756 (1999): 911-914.

Power-laws can have very different consequences from Gaussian distributions.

In particular they allow for rare events.




Arron Clauset (UNM CS Phd, now Univ Colorado Boulder)
Provides more examples and a critical analysis.

[Clauset, A. Power-law Distributions in Empirical Data, 2009]

$$\lambda(\xi) = \xi^{-(1+\alpha)} \qquad 0<\alpha< 2$$